MATH 261 EXAM II PRACTICE PROBLEMS

These practice problems are pulled from actual midterms in previous semesters. Exam 2 typically has 6 problems on it, with no more than one problem of any given type (e.g., don't expect two problems about linearization). Also, **please be aware** that this is not intended as a comprehensive list of all possible problem types! In other words, you are responsible for all topics covered during the period leading up to this exam, whether they are represented in this list or not. See your notes and the suggested homework for a comprehensive list.

1. (a) Compute $\displaystyle\lim_{(x,y,z)\to(1,0,1)} \frac{\sin(y)+\ln(xz)}{x^2+y^2+z^2}$.

   (b) Show that the function $f(x,y) = \dfrac{(x-1)^2+y^2}{(x-1)^2+2y^2}$ has no limit as (x,y) approaches $(1,0)$.

2. Evaluate the limit or show that the limit does not exist. Proper work must be shown.

   (a) $\displaystyle\lim_{(x,y)\to(0,\frac{\pi}{2})} \frac{\sin(y)\cos(y)}{ye^x}$

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+3y^4}$

3. (a) Let $f(x,y,z) = x^2y + z^2 + 8$ and $P = (2,1,0)$. Find the unit vector which points in the direction of most rapid increase of $f(x,y,z)$ at P.

   (b) Suppose $\nabla g = \langle y - x, x^2 - y\rangle$. Find all directions for which the directional derivative of g at the point $(1,3)$ is 2. Be sure to express your answer(s) as unit vector(s).

4. Suppose $\nabla f = \langle 3x^2 + 4y, \frac{5}{2}x + 16y\rangle$ for some function $f(x,y)$ and let P be the point $(2,0)$.

   (a) Compute the (unit length) direction of greatest increase at P.

   (b) Compute the directional derivative of $f(x,y)$ at point P in the direction found in part (a).

   (c) Find all (unit length) directions in which the directional derivative at P is 0.

5. Let $f(x,y) = \cos(x) + \sin(y)$. Give an equation for the plane tangent to the graph of $z = f(x,y)$ at the point $P = \left(\frac{\pi}{2}, \pi, 0\right)$. Please simplify your solution to the form $Ax + By + Cz = D$.

6. Consider the surface given by $F(x,y,z) = xy^2z + \ln(xy) - 1 = 0$. Find the equation of the tangent plane at $(e,1,0)$. Please simplify your solution to the form $Ax+By+Cz = D$.

7. Consider the function

$$F(x,y,z) = \sin(x) + \cos(y) + e^z.$$

(a) Evaluate ∇F at the point $(\pi, 0, \ln 2)$.

(b) Find equations for the normal line to $F(x, y, z) = 1$ at $(\pi, 0, \ln 2)$ in parametric form.

8. (a) Find (but do **not** classify) all critical points of $f(x, y) = x^2 - \frac{1}{18}(x-1)y^2$.

   (b) The function $g(x, y) = x^3 + x^2 + 3y^2 - x - 12y + 11$ has critical points $(-1, 2)$ and $(\frac{1}{3}, 2)$. Classify these two points as saddle points, local mins, or local maxes of $g(x, y)$, clearly indicating any computed values used in your classification.

9. Consider the function $f(x, y) = x^3 + 3x^2 + y^2 + 2y$. Fill in the four empty boxes in the following table. Each row corresponds to one of the two critical points. The columns provide the point, the value of $f_{xx}f_{yy} - f_{xy}^2$ at the point, and the interpretation of that value (local min, local max, or saddle point).

Point	$f_{xx}f_{yy} - f_{xy}^2$	MIN/MAX/SP
$(0, -1)$		
	-12	

10. Consider the function $f(x, y) = x^3 + 8y^3 - 12xy$.

    (a) $(0, 0)$ is one critical point. Classify it as a min, a max, or a saddle point. Clearly identify the value of $f_{xx}f_{yy} - f_{xy}^2$ and the second value needed to distinguish between a max and a min (if needed).

    (b) Find and classify all remaining critical points, clearly identifying values necessary to your classification, as in the previous part.

11. Suppose we want to find the point on the plane $x + 2y + 3z = 4$ closest to the point $(1, 1, 2)$. Write the system of equations you would need to solve to use Lagrange multipliers for this problem, but **DO NOT** solve the system.

12. Using Lagrange multipliers, find the point on the line $x + 2y = 5$ nearest the origin.

13. (a) Consider the function $f(x, y) = e^x + \sin(y)$ and point $P = (0, \pi)$. Give the linearization of $f(x, y)$ at P.

    (b) Suppose that a function $g(x, y)$ has been linearized by $L(x, y)$ at point $(2, 3)$ over the rectangle given by $|x - 2| \leq 0.1$, $|y - 3| \leq 0.2$. If $f_{xx} = 1$, $f_{xy} = -2$, and $f_{yy} = 12y - 36$, what is the best possible upper bound on this error, using the method described in the book and in class? (You may leave your answer as a numerical expression, i.e., not simplified down to a number.)

14. (a) Suppose we approximate a function $g(x, y)$ by its linearization at point $(1, 3)$. Provide the best upper bound you can for the error in this approximation, over the box $|x - 1| \leq 0.1$, $|y - 3| \leq 0.1$, given that

$$g_{xx} = 3(x-1), g_{xy} = 0, g_{yy} = y - 3.$$

You need not multiply out your answer; you may leave your answer as a numerical expression.

(b) Give the linearization $L(x, y)$ of $f(x, y) = \ln(x)\cos(y)$ at the point $(x, y) = (1, \pi)$.

15. (a) Find the linearization of $f(x, y) = x^2y + y$ at $P(1, 2)$. Do NOT simplify beyond the form $L(x, y) = a + b(x - x_0) + c(y - y_0)$.

(b) Using the method from class, find the lowest possible upper bound for $|E|$, the error in the approximation of $f(x, y)$ with $L(x, y)$ over the rectangle $|x-1| \leq 0.1$, $|y - 2| \leq 0.2$.

16. (a) For a function $f(x, y, z)$ with $x = x(r, t), y = y(r, t), z = z(r, t)$, what is the general multivariate chain rule for computing $\partial f/\partial r$?

(b) Let

$$\begin{aligned} f(x, y, z) &= x^2 + 2yz + 3z^2, \\ x(r, t) &= r^2 + \cos(\ln t), \\ y(r, t) &= e^{t-1} \\ z(r, t) &= r^2 + t^2. \end{aligned}$$

Evaluate $\partial f/\partial r$ at the point $(r, t) = (1, 1)$.

17. (a) For a function $f(r, s)$ with $r = r(x, y, z)$ and $s = s(x, y, z)$, what is the general multivariate chain rule for computing $\frac{\partial f}{\partial y}$?

(b) Given $f = rs + s^2, r = xy^2, s = \cos(yz)$, evaluate $\frac{\partial f}{\partial y}$ at the point $(x, y, z) = (e, \frac{\pi}{2}, 4)$.

SPOILER ALERT: Solutions start on the next page.

SOLUTIONS

WARNING: These solutions are not fully justified. Be sure to provide full justification with your solutions (especially where this is explicitly requested) so that we may provide partial credit, where applicable. If you are having trouble getting these answers, please come to office hours and/or exam review sessions. See the course website for details. Of course, if there seems to be an error with these solutions (which is very possible!), please let an instructor or the coordinator know.

1. (a) 0
   (b) $\frac{1}{2}$ along $x = 1$, 1 along $y = 0$. Alternatively, using $y = m(x-1)$, $\lim = \frac{1+m^2}{1+2m^2}$.
2. (a) 0
   (b) DNE: 0 along $x = 0$, but $\frac{1}{4}$ along $y = x$
3. (a) $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right\rangle$
   (b) $\langle 1, 0\rangle$, $\langle 0, -1\rangle$
4. (a) $\langle \frac{12}{13}, \frac{5}{13} \rangle$
   (b) 13
   (c) $\langle \frac{5}{13}, \frac{-12}{13} \rangle$, $\langle \frac{-5}{13}, \frac{12}{13} \rangle$
5. $x + y + z = \dfrac{3\pi}{2}$
6. $\frac{1}{e}x + y + ez = 2$
7. (a) $\langle -1, 0, 2\rangle$
   (b) $x = \pi - t, y = 0, z = \ln(2) + 2t$
8. (a) $(0, 0), (1, 6), (1, -6)$
   (b) $(-1, 2)$ is a saddle point (value: -36); $(\frac{1}{3}, 2)$ is a minimum (values: 12, 2)
9.

Point	$f_{xx}f_{yy} - f_{xy}^2$	MIN/MAX/SP
$(0, -1)$	12	min
$(-2, -1)$	-12	saddle point

10. (a) saddle point (value: -144)
    (b) $(0, 0)$ is a min (values: 432, 12)
11. $x - 1 = \lambda$, $y - 1 = 2\lambda$, $z - 2 = 3\lambda$

12. $(1,2)$

13. (a) $L(x,y) = 1 + x - y + \pi$
    (b) $\frac{1}{2}(2.4)(0.09)$ (so $M = 2.4$)

14. (a) $\frac{1}{2}(0.3)(0.2)^2$
    (b) $L(x,y) = 0 + (-1)(x-1) + 0(y-\pi)$

15. (a) $L(x,y) = 4 + 4(x-1) + 2(y-2)$
    (b) $\frac{1}{2}(4.4)(0.3)^2$

16. (a) $\dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial r}$
    (b) 36

17. (a) $\dfrac{\partial f}{\partial y} = \dfrac{\partial f}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial f}{\partial s}\dfrac{\partial s}{\partial y}$
    (b) $e\pi$