

# Math 261 Exam 1 Review Formulas & Reminders

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Here are a few formulas that might be handy for Exam 1. You *cannot* bring this to the exam, but hopefully it helps with studying...

**WARNING:** I do not guarantee that this is a comprehensive list! Also, please note that there are various alternative formulations for some of these formulas – I am just picking those that I like the best. Finally, there could be typos – beware!

- (Vector between two points) Given points  $(p_1, p_2, p_3)$  and  $(q_1, q_2, q_3)$  in  $\mathbb{R}^3$ , the vector between them is just  $Q - P = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3 \rangle$ .
- (Length of a vector)  $|\mathbf{v}| = |\langle v_1, v_2, v_3 \rangle| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ .
- (Make a vector unit length) Just divide the vector by its length:  $\frac{\mathbf{v}}{|\mathbf{v}|}$ .
- (Dot product)  $\mathbf{v} \cdot \mathbf{w} = \langle v_1, v_2, v_3 \rangle \cdot \langle w_1, w_2, w_3 \rangle = v_1w_1 + v_2w_2 + v_3w_3$  (a number!) Recall that  $\mathbf{v}$  and  $\mathbf{w}$  are orthogonal if and only if  $\mathbf{v} \cdot \mathbf{w} = 0$ .
- (Another dot product formula)  $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos \theta$ , where  $\theta$  is the angle between  $\mathbf{u}$  and  $\mathbf{v}$ .
- (Projection) The projection of vector  $\mathbf{u}$  onto vector  $\mathbf{v}$  is  $\text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}$ .  
The scalar component of the projection is  $\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|}$  (since  $\frac{\mathbf{v}}{|\mathbf{v}|}$  is the unit-length direction).
- (Cross product)  $\mathbf{v} \times \mathbf{w} = \langle v_1, v_2, v_3 \rangle \times \langle w_1, w_2, w_3 \rangle = \langle v_2w_3 - v_3w_2, -(v_1w_3 - v_3w_1), v_1w_2 - v_2w_1 \rangle$  (a vector!). Don't forget to negate that middle coordinate! (It might be easier to remember the 3 Xs way I taught you to compute the cross product.) Recall that two nonzero vectors are parallel if and only if their cross product is the zero vector.
- (Area of a triangle) The area of a triangle with edges  $\mathbf{v}$  and  $\mathbf{w}$  is  $\frac{|\mathbf{v} \times \mathbf{w}|}{2}$ . The volume of a parallelepiped with edges  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  is  $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ .
- (Another cross product formula)  $\mathbf{v} \times \mathbf{w} = (|\mathbf{v}||\mathbf{w}| \sin \theta) \mathbf{n}$  where  $\theta$  is the angle between  $\mathbf{v}$  and  $\mathbf{w}$  and  $\mathbf{n}$  is a unit vector in the normal direction (orthogonal to  $\mathbf{v}$  and  $\mathbf{w}$ ).
- (Equations for a line) Given a point  $P = (p_1, p_2, p_3)$  on a line and a vector  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$  in the direction of the line, the following are parametric equations for the line:

$$x(t) = p_1 + tv_1$$

$$y(t) = p_2 + tv_2$$

$$z(t) = p_3 + tv_3,$$

which could also be written more succinctly as  $\mathbf{r}(t) = \langle p_1 + tv_1, p_2 + tv_2, p_3 + tv_3 \rangle$ . (Parameterizations are not unique - they depend on the choices of  $P$  and  $\mathbf{v}$ .)

- (Distance from point to line) Given point  $S$  and a line with direction  $\mathbf{v}$  and a point (any point)  $P$ , the distance from  $S$  to the line is  $\frac{|\mathbf{PS} \times \mathbf{v}|}{|\mathbf{v}|}$ , where  $\mathbf{PS}$  is the vector from  $P$  to  $S$ . If the line is given to you in parametric form, you can find a point on the line by plugging in any value of  $t$ , e.g.,  $t = 0$ .
- (Equation for a plane) A plane is given by a normal vector  $\langle A, B, C \rangle$  and a point  $(x_0, y_0, z_0)$  on the plane. The equation is then  $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$ . One way to get the normal is to take the cross product of two vectors in the plane (that have the same initial point).

- (Line of intersection of two planes) Given two planes with normal vectors  $\mathbf{n}_1$  and  $\mathbf{n}_2$ , respectively, the vector  $\mathbf{n}_1 \times \mathbf{n}_2$  points in the direction of the line of intersection of the two planes (assuming they intersect!). To get a point on this line, you can solve the system of two plane equations. Note that there will be 3 variables and 2 equations in this linear system, so you should just set one of the variables to a constant, e.g.,  $x = 0$ , and solve for the other two. (Two planes are parallel – don't intersect – if they have parallel normals, i.e.,  $\mathbf{n}_1 \times \mathbf{n}_2 = \mathbf{0}$ .)
- (Distance from point to plane) Given point  $S$  and a plane with normal  $\mathbf{n}$  and point  $P$  on the plane, the distance from  $S$  to the plane is  $\frac{\mathbf{PS} \cdot \mathbf{n}}{|\mathbf{n}|}$ .
- (Angle between planes) The angle between two planes is the angle between their normal vectors.
- (Position, velocity, and acceleration) If  $\mathbf{r}(t)$  represents the position of a particle, then  $\mathbf{v}(t) = \mathbf{r}'(t)$  is the velocity and  $\mathbf{a}(t) = \mathbf{v}'(t)$  is the acceleration. If you are given an initial value problem (e.g.,  $\mathbf{a}(t)$  and the values of  $\mathbf{v}(t)$  and  $\mathbf{r}(t)$  at some point), you integrate to work your way up to  $\mathbf{r}(t)$ , using the values of  $\mathbf{v}(t)$  and  $\mathbf{r}(t)$  to find the constants of integration.
- (Decomposing acceleration) Acceleration along a curve can be written as a linear combination of the tangent and normal directions. In particular:

$$\mathbf{a}(t) = a_T(t)\mathbf{T}(t) + a_N(t)\mathbf{N}(t),$$

where:

- $\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|}$ ,
- $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$  (this is not the easy way to compute this!),
- $a_T = \mathbf{a} \cdot \mathbf{T}$ , and
- $a_N$  can be found from the Pythagorean identity  $|\mathbf{a}|^2 = a_T^2 + a_N^2$ .

If you need to compute all of these, the typical strategy is to compute  $\mathbf{v}$  and  $\mathbf{a}$ , then compute  $\mathbf{T}$  as above, then  $a_T$ , then  $a_N$ . Finally, you can find  $\mathbf{N}$  using the fact that  $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ .