MATH 561: Numerical Analysis I

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Homework assignment 6 – due 04/25/2017

Problem 1 (Constraint qualification). We discussed in class that solutions of the equality-constrained optimization problem

$$\begin{aligned} \min\ & f(x_1,\dots,x_n) \\ \text{such that } & g_1(x_1,\dots,x_n) = 0, \\ & \vdots, \\ & g_{n_e}(x_1,\dots,x_n) = 0 \end{aligned}$$

where $n_e < n$ are points where the following conditions have to hold:

$$\begin{aligned} \nabla f(x) - \sum_{i=1}^{n_e} \lambda_i \nabla g(x) &= 0, \\ g_1(x_1,\dots,x_n) &= 0, \\ &\vdots \\ g_{n_e}(x_1,\dots,x_n) &= 0. \end{aligned}$$

It was also, very briefly, mentioned that this is only true "under certain conditions".

Consider the following two problems in two variables with one constraint:

$$\begin{aligned} \min\ & x_1^2 + x_2^2 \\ \text{such that } & (x_1-1)^2 = 0, \end{aligned}$$

and

$$\begin{aligned} \min\ & x_1^2 + 100x_2^2 \\ \text{such that } & (5x_1 - x_2 - 1)(5x_1 + x_2 - 1) = 0. \end{aligned}$$

Visualize these two problems and find their solutions analytically. Verify in both cases that the optimality conditions stated above do not hold. What is going wrong here? **(20 points)**

Problem 2 (The quadratic penalty method). The quadratic penalty method converts an optimization problem with equality constraint into an unconstrained problem. Take the particularly simple problem

$$\begin{aligned} \min\ & x^2 \\ \text{such that } & x - 1 = 0. \end{aligned}$$

This problem is of course silly because there is only a single point ($x = 1$) that actually satisfies the constraint, and so it doesn't really matter what function we try to minimize. But we can at least pretend that this is a problem we care about.

The quadratic penalty method seeks a minimizer x^*_μ of a function Q_μ that is derived from the problem above. We then want to let $\mu \to 0$ and see where x^*_μ converges to.

This problem is simple enough that we can find x^*_μ analytically. What is Q_μ here? Show Q_μ for a few values of μ. As a function of μ, what is x^*_μ, and what does it converge to? Is it the solution of the original problem?

(15 points)

Problem 3 (The quadratic penalty method). Consider the following two-dimensional problem with one constraint:

$$\min\ x_1^2 - x_2^2$$
$$\text{such that } x_1^2 + x_2^2 - 1 = 0.$$

Note that the there would be no solution without the constraint because the objective function is not bounded from below.

Write a program that finds the solution of this problem using the quadratic penalty method. **(20 points)**

Problem 4 (Linear-quadratic programs). "Linear-quadratic programs" are optimization problems in which the objective function is a quadratic function and the constraints are linear (or, rather affine). An example is the following problem:

$$\min\ (x_1 - x_2)^2 + (x_1 + x_2)^2$$
$$\text{such that } x_1 + x_2 - 5 = 0.$$

As in the previous problem, the objective function is not bounded from below and so we need the constraint for the problem to have a solution.

Use the method of Lagrange multipliers to define a Lagrangian, and then find the solution of the problem computationally by solving the equations that define a stationary point of the Lagrangian. Verify graphically or analytically that this is indeed the solution of the original problem. **(20 points)**

Problem 5 (Sequential quadratic programming). Implement a program that applies the sequential quadratic programming method to find the solution of the following problem:

$$\min\ e^{x_1} + e^{2x_2}$$
$$\text{such that } x_1^2 + x_2^2 - 1 = 0.$$

(25 points)