MATH 561: Numerical Analysis I

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Homework assignment 3 – due 2/28/2017

Problem 1 (Convergence of Richardson iteration). Richardson iteration is another variant of the methods presented. It can be written in the form

$$x^{k+1} = x^k - \omega\big(Ax^k - b\big),$$

where $\omega > 0$ is called the damping factor. Follow these steps to prove that this method yields a contraction with respect to the norm $\|.\|_2$ if A is a symmetric positive definite matrix and $0 < \omega < 1/\lambda_{\max}$, where $\lambda_{\max}$ is the largest eigenvalue of A.

a) State what the operator $T = \mathbf{1} - BA$ of the iteration is, by identifying what B is for the iteration above.

b) Determine the eigenvalues (and if necessary eigenvectors) of this matrix T.

c) In class we saw that an iterative scheme converges if $\|I - BA\| \leq \delta < 1$. Use the decomposition of part b) to show this property in the l_2 matrix norm, i.e. to show that $\|I - BA\|_2 < 1$. For this remember that if T is a positive definite, symmetric matrix, then $\|T\|_2 = \lambda_{\max}(T)$, where $\lambda_{\max}(T)$ denotes the largest eigenvalue of the matrix T. **(10 points)**

Problem 2 (Condition numbers). Calculate the condition numbers $\kappa(A) = \|A\|\,\|A^{-1}\|$ with respect to the l_1, l_∞ and l_2 norms for the matrix

$$A = \begin{pmatrix} 1 & 1.001 \\ 0.999 & 1 \end{pmatrix}.$$

(10 points)

Problem 3 (Error propagation). With the matrix from Problem 2, consider the solutions $x, \tilde{x}$ of the following linear systems:

$$Ax = b, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$A\tilde{x} = \tilde{b}, \qquad \tilde{b} = \begin{pmatrix} 1 \\ 1.001 \end{pmatrix}.$$

(Imagine the former to be the exact right hand side, and the latter to be one that is contaminated by measurement uncertainty, statistical error, etc.)

Solve for x and $\tilde{x}$. Calculate the relative difference in the right hand side $\epsilon_r = \|b - \tilde{b}\|/\|b\|$ and the relative error $e_r = \|x - \tilde{x}\|/\|x\|$ in the solution, each for both the l_2 and the l_∞ norm.

Using your result from Problem 3, do ϵ_r and e_r satisfy the estimates discussed in class, i.e. that $e_r \leq \kappa(A)\epsilon_r$? **(10 points)**

Problem 4 (Matrix norms). Prove that if $\|A\|$ is a matrix norm induced by a vector norm $\|v\|$, then $\|I\| = 1$ where I is the identity matrix.

We have shown that induced matrix norms indeed satisfy the three norm conditions. However, there may be other matrix norms that are not induced and that nevertheless also satisfy the norm conditions. Do you think there can be matrix norms that satisfy the norm conditions but for which $\|I\| \neq 1$? If so, give an example. **(5 points)**

Problem 5 (Jacobi iteration). Take the same system of linear equations from last homework's Problem 7, i.e., let A, b be the 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases} \qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Use a method of your choice to compute the solution $x = A^{-1}b$ of this problem and store it in a vector. It is not important what method you use – you are welcome to use Matlab or any other system to just solve the linear problem and obtain x.

Next implement the Jacobi iteration, e.g., using any implementation you already have from the last homework. Run the Jacobi iteration on this problem, starting with the vector $x^{(0)} = 0$. Because for this (relatively small and simple) case you know the exact solution x, you can compute the error $\|x^{(k)} - x\|_{l_2}$ in each iteration. It should, of course, decrease in each iteration compared to the previous one.

Produce a graph in which you plot the error $\|x^{(k)} - x\|_{l_2}$ against the iteration k. You will want to use a logarithmic y-axis to better show how the error decreases over time. **(20 points)**

Problem 6 (Gauss-Seidel and Symmetric Gauss-Seidel iterations). Repeat problem 5, but showing results for both the Gauss-Seidel and the Symmetric Gauss-Seidel iteration instead of the Jacobi iteration.

(20 points)

Problem 7 (Steepest Descent iteration). Repeat problem 5 one more time, but with the Steepest Descent method for solving linear problems. As discussed in class, this method can be thought of as a variation of the Richardson iteration in which we choose the factor ω differently in each iteration, rather than choosing it fixed once and for all. **(15 points)**

Problem 8 (Comparing methods). Discuss which of the methods you tried in Problems 5–7 converge faster or slower. Also compare the work that you need to do in each iteration for all of these iterative methods. Which method would you rate "best" on this problem? **(10 points)**