MATH 689: Numerical Optimization

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Homework assignment 11 – due Tuesday 4/15/2014

Problem 1 (Solution operator). Consider the problem of throwing a ball in a two-dimensional x–z world (i.e., one horizontal and one vertical direction). If its initial position and velocity are $\mathbf{x}_0, \mathbf{v}_0$, both two-dimensional vectors, then the ball's trajectory $\mathbf{x}(t)$ will satisfy the following differential equation:

$$\begin{aligned} m\ddot{\mathbf{x}}(t) &= m\mathbf{g}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \dot{\mathbf{x}}(0) &= \mathbf{v}_0. \end{aligned}$$

Here, $\mathbf{g} = (0, -9.81\text{m/s}^2)^T$ is the direction of gravity (straight down).

For this problem, you can integrate the equations analytically and come up with an explicit expression for $\mathbf{x}(t)$. Using this expression, state the form of the solution operator $S(\mathbf{x}_0, \mathbf{v}_0, 0, t)$. **(2 points)**

Problem 2 (A simple optimal control problem, part 1). Now consider throwing the ball with an aim, for example if you are a doing a free throw in basketball. To this end, the ball's initial position is $\mathbf{x}_0 = (0, 1.8\text{m})^T$ and you want the ball to go through the net at $\hat{\mathbf{x}} = (4.2\text{m}, 3\text{m})^T$. A player's mind then needs to solve an optimal control problem that can be stated as follows to get the ball as close as possible to the center of the hoop:

$$\begin{aligned} \min_{\mathbf{v}_0, T} \quad & \frac{1}{2}\|\mathbf{x}(T) - \hat{\mathbf{x}}\|^2 \\ m\ddot{\mathbf{x}}(t) &= m\mathbf{g}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \dot{\mathbf{x}}(0) &= \mathbf{v}_0. \end{aligned}$$

Using the solution operator, eliminate the differential equation constraints and reformulate the problem exclusively in terms of the optimization variables $\{\mathbf{v}_0, T\} \in \mathbb{R}^3$.

Plot this reformulated objective function as a function of three variables. How many minima are there? **(4 points)**

Problem 3 (A simple optimal control problem, part 2). Humans actually work differently: our body typically wants to minimize the energy expended for something. This suggests the following problem instead:

$$\begin{aligned} \min_{\mathbf{v}_0, T} \quad & \frac{1}{2} m \|\mathbf{v}_0\|^2 \\ m\ddot{\mathbf{x}}(t) &= m\mathbf{g}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \dot{\mathbf{x}}(0) &= \mathbf{v}_0, \\ \mathbf{x}(T) - \hat{\mathbf{x}} &= 0. \end{aligned}$$

You can again eliminate the three constraints related to the differential equation and its initial conditions using the solution operator. The last constraint (actually two, for the two components of $\mathbf{x}(T)$) remains. However, you can use it to eliminate two of the three optimization variables.

Do this reformulation to state the problem in terms of only a single variable (e.g., the flight time T, or one of the two components of $\mathbf{v}_0$). Plot the objective function and determine its minimum. Plot the trajectory the ball takes. What is the angle at which the ball is thrown? (90 degrees would be vertically up, 0 degrees would be horizontal. For comparison, most good free throw shooters throw at around 52 degrees.)

(6 points)

Problem 4 (A more realistic optimal control problem). Reconsider the previous problem, but let us include air friction into the equation:

$$\begin{aligned} \min_{\mathbf{v}_0, T} \quad & \frac{1}{2} m \|\mathbf{v}_0\|^2 \\ m\ddot{\mathbf{x}}(t) &= m\mathbf{g} - C\|\dot{\mathbf{x}}(t)\|\dot{\mathbf{x}}(t), \\ \mathbf{x}(0) &= \mathbf{x}_0, \\ \dot{\mathbf{x}}(0) &= \mathbf{v}_0, \\ \mathbf{x}(T) - \hat{\mathbf{x}} &= 0. \end{aligned}$$

Use $m = 0.62\text{kg}$ and $C = \frac{1}{2}\rho C_d A$ with the density of air $\rho = 1.2\text{kg/m}^3$, the coefficient of drag $C_d = 0.5$ and a basketball's area $A = 0.046\text{m}^2$.

In this example, you will likely not be able to find an explicit solution of the differential equation, but you can use a numerical approximation of the solution operator. This gives you a way to solve the problem numerically where the objective function only contains $\mathbf{v}_0$ but the last constraint is a nonlinear function of the optimization variables through the solution operator.

Find the solution to this problem using your favorite solver for equality constrained optimization problems.

(6 points)

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!