MATH 689: Numerical Optimization

Prof. Wolfgang Bangerth // Blocker 507D // `bangerth@math.tamu.edu`

Homework assignment 7 – due Tuesday 3/18/2014

Problem 1 (Lagrange multipliers and Lagrangians). Consider the following, somewhat boring, equility-constrained problem over $x \in \mathbb{R}$:

$$\begin{aligned} \min_x \quad & x^2 \\ & x - 1 = 0. \end{aligned}$$

Answer the following questions: (i) Does it have a solution? (ii) Is the solution unique? (iii) Do the LICQ conditions hold? (iv) Can we derive necessary conditions for the solution that contain a Lagrange multiplier, and if so how do they look? (v) What is the solution to the problem above, and what is the value of the Lagrange multiplier if one exists? (vi) Verify that the Lagrangian of this problem describes a quadratic function in x, λ that has only a single, unique stationary point equal to the solution of the constrained minimization problem. (vii) Verify that the matrix of second derivatives of the Lagrangian is indefinite, i.e. that the stationary point is a saddle point, not a minimum or maximum. For the last two parts, it may be useful to plot the Lagrangian as a function of x and λ. **(4 points)**

Problem 2 (Quadratic programming). Numerically find the solution of the following problem in $x \in \mathbb{R}^4$:

$$\begin{aligned} \min_{x\in\mathbb{R}^4} \quad & \frac{1}{2}x^T G x + d^T x + e \\ & Ax - b = 0, \end{aligned}$$

where

$$G = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 4 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \qquad e = 0,$$

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Explain how the solution and the values of the Lagrange multipliers are going to change if the constraint matrix was changed to

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 6 \end{pmatrix}.$$

(4 points)

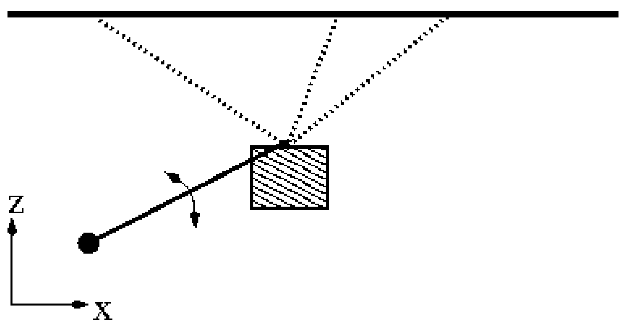
Problem 3 (SQP). Use the SQP method with full step length to find the solution of the following problem:

$$\min_{x\in\mathbb{R}^2} \quad -\frac{1}{1+x_1^2+x_2^2}$$
$$x_1+x_2-1=0$$

You may have to start close enough to the solution of this problem to converge. (As always, you will benefit if you plot the objective function and constraint beforehand, as it gives you some intuition on where you expect the solution to lie and a way to confirm that what the algorithm gives you makes sense.)

(3 points)

Problem 4 (A modeling exercise, revisited once more). Consider the following system of three springs suspended from the ceiling at positions $(x,z)=(-20\text{cm},0\text{cm})$, $(5\text{cm},0\text{cm})$ and $(15\text{cm},0\text{cm})$ and a rod of fixed length 20cm that is attached at one end at $(-25\text{cm},-20\text{cm})$ and at the other end to the same point where the springs meet each other:



Each spring has a rest length of $L_0 = 20\text{cm}$, and extending (or compressing) spring $i, i=1\dots3$ to a length L_i requires an energy of $E_{\text{spring},i} = \frac{1}{2}D(L_i-L_0)^2$ where the spring constant for all three springs equals $D=300\frac{N}{m}$. On the other hand, the potential energy of the body is $E_{\text{pot}} = mgz$ where the body's mass is $m=500g$, the gravity constant is $g=9.81\frac{\text{m}}{\text{s}^2}$, and z is the vertical coordinate of the body's position.

Use the method of Lagrange multipliers to derive necessary conditions for the solution of this problem. Solve this system of nonlinear equations using Newton's method – in other words, apply the SQP method to the constrained optimization problem. In your answer, show this location, the value of the total energy function at this location, and the value of the Lagrange multiplier. Is there only a single minimum to the constrained problem?

(6 points)

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!