MATH 437: Principles of Numerical Analysis

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Homework assignment 11 – due Thursday 11/21/2013

Problem 1 (Numerical solution of a scalar ODE). Consider the following scalar ordinary differential equation (ODE):

$$x'(t) = \frac{1}{3x(t)^2},$$
$$x(0) = \frac{1}{10^{1/3}}.$$

The solution of this equation is $x(t) = (t + \frac{1}{10})^{1/3}$.

Compute approximations to $x(4)$ using the

- first order Taylor expansion method,
- second order Taylor expansion method,
- implicit Euler method,

each with step sizes $h = 2, 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{32}$. Compute their respective errors $e = |x_N - x(4)|$ where x_N is the approximation to $x(4)$ at the end of the last time step, and compute the convergence rates. Compare the accuracy of all these methods for the same step size h. **(8 points)**

Problem 2 (Numerical solution of a vector-valued ODE). A rocket that is shot up vertically experiences upward acceleration from its engines, and downward acceleration due to gravity. Its height therefore satisfies Newton's law

$$d''(t) = \frac{F(t)}{m(t)}, \tag{1}$$

where $d(t)$ denotes the distance from the earth's center. Assume that the rocket is initially at rest at $d(0) = 6371000$ (in meters). After ignition, the engines produce a constant thrust for 10 minutes before shutting down:

$$T(t) = \begin{cases} 12 & \text{for } t < 600, \\ 0 & \text{for } t \geq 600. \end{cases}$$

On the other hand, gravity generates the force

$$G(t) = -(6371000)^2 \frac{10m(t)}{d(t)^2}.$$

The total force is $F(t) = T(t) + G(t)$. The mass of the rocket decreases while fuel is burnt in the engines according to

$$m(t) = \begin{cases} 1 - \frac{0.9t}{600} & \text{for } t < 600, \\ 0.1 & \text{for } t \geq 600. \end{cases}$$

Compute the altitude of the rocket for times between $t = 0$ and $t = 36000$ using the explicit Euler method. Try to determine the altitude up to an accuracy of 100 meters by choosing an appropriate time step size. **(5 points)**

Problem 3 (Some parameter determination with ODEs). In skydiving, freefall is a balance between the force gravity exerts on the skydiver, and the counteracting air friction. Within the atmosphere, gravity does not depend on the altitude (and is approximately $10ms^{-2}$), and air friction increases like the square of the velocity. We can therefore describe the falling velocity by the ODE

$$v'(t) = 10 - av(t)^2,$$
$$v(0) = 0.$$

Using your own ODE solver, compute an approximate value for the coefficient a such that the speed of the skydiver after 10 seconds is $v(10) = 50$ (that's a realistic free fall velocity in meters per second: approximately 115 mph).

(For this question, creativity in finding a way to approximate a is encouraged—the way counts, not the result up to 6 digits.) **(3 points)**