Homework assignment 10 – due Thursday 11/14/2013

Problem 1 (Finite difference approximation of the derivative). Take the function defined by

\[ f(x) = \begin{cases} 
\frac{1}{2}x^3 + x^2 & \text{for } x < 0 \\
\frac{x}{3} & \text{for } x \geq 0.
\end{cases} \]

Compute a finite difference approximation to \( f'(x_0) \) at \( x_0 = 1 \) with both the one-sided and the symmetric two-sided formula. Use step sizes \( h = 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{64} \). Determine experimentally the convergence orders you observe as \( h \to 0 \).

Repeat these computations for \( x_0 = 0 \). What convergence orders do you observe? Why? (4 points)

Problem 2 (Derivatives of an implicit function). Let \( f(x) \) be defined implicitly as follows: for every \( x > 0 \), \( f(x) \) is that value \( y \) for which \( ye^y = x \). In other words, every time one wants to evaluate \( f(x) \) for a particular value \( x \), one has to solve equation (1) for \( y \). This can be done using Newton’s method, for example, or any of the other root finding algorithms we had in class applied to the function \( g(y) = ye^y - x \).

Plot \( f(x) \) for \( 0 \leq x \leq 10 \). Compute an accurate approximation to \( f'(2) \). (4 points)

Problem 3 (Integration of an implicit function). Let \( f(x) \) be defined as in Problem 2. Compute

\[ \int_0^{10} f(x) \, dx \]

using both the box rule as well as the trapezoidal rule for step sizes \( h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{32} \). Determine the order of convergence for both methods. (4 points)

Problem 4 (A proof). In last week’s homework, you were asked to find the \( l_\infty \) best approximating linear function to 10 data points. Let’s simplify the situation a little bit: assume we had only wanted to find a constant best
approximation, i.e. a function \( p_0(x) = c_0 \), to all these data points. Then, this involves finding the coefficient \( c_0^* \) for which the function

\[
e(c_0) = \max_{1 \leq i \leq N} |c_0 - y_i|
\]

takes on its minimum. If one wants to phrase this differently, one could also say that we are looking for the optimal coefficient \( c_0^* \) for which

\[
e(c_0^*) = \min_{c_0} \max_{1 \leq i \leq N} |c_0 - y_i|.
\]

One could wonder if there is indeed only a single such value \( c_0 \), or if it may be possible to have a number of different values for \( c_0 \) for which the corresponding functions \( p_0(x) = c_0 \) are all best approximations to the data points.

Prove that the function \( e(c_0) \) defined above has only a single minimum and that consequently there is exactly one, well-defined best \( l_\infty \) approximation among the constant functions \( p_0(x) = c_0 \). (Hint: Try your hands first on the case where there are only two data points, i.e. \( e(c_0) = \max\{|c_0 - y_1|, |c_0 - y_2|\} \) and then generalize to \( N \) data points.)

Comment on what happens if you were looking for linear approximations \( p_1(x) = c_0 + c_1 x \) with corresponding error function

\[
e(c_0, c_1) = \max_{1 \leq i \leq N} |c_0 + c_1 x_i - y_i|.
\]

Does this two-dimensional function have a single, unique minimum as well?

(4 points)