MATH 437: Principles of Numerical Analysis

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Homework assignment 10 – due Thursday 11/14/2013

Problem 1 (Finite difference approximation of the derivative). Take the function defined by

$$f(x) = \begin{cases} \frac{1}{2}x^3 + x^2 & \text{for } x < 0 \\ x^3 & \text{for } x \geq 0. \end{cases}$$

Compute a finite difference approximation to $f'(x_0)$ at $x_0 = 1$ with both the one-sided and the symmetric two-sided formula. Use step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{64}$. Determine experimentally the convergence orders you observe as $h \to 0$.

Repeat these computations for $x_0 = 0$. What convergence orders do you observe? Why? **(4 points)**

Problem 2 (Derivatives of an implicit function). Let $f(x)$ be defined implicitly as follows: for every $x > 0$, $f(x)$ is that value y for which

$$ye^y = x. \tag{1}$$

In other words, every time one wants to evaluate $f(x)$ for a particular value x, one has to solve equation (1) for y. This can be done using Newton's method, for example, or any of the other root finding algorithms we had in class applied to the function $g(y) = ye^y - x$.

Plot $f(x)$ for $0 \leq x \leq 10$. Compute an accurate approximation to $f'(2)$.

(4 points)

Problem 3 (Integration of an implicit function). Let $f(x)$ be defined as in Problem 2. Compute

$$\int_0^{10} f(x)\, dx$$

using both the box rule as well as the trapezoidal rule for step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{32}$. Determine the order of convergence for both methods.

(4 points)

Problem 4 (A proof). In last week's homework, you were asked to find the l_∞ best approximating linear function to 10 data points. Let's simplify the situation a little bit: assume we had only wanted to find a *constant* best approximation, i.e. a function $p_0(x) = c_0$, to all these data points. Then, this involves finding the coefficient c_0^* for which the function

$$e(c_0) = \max_{1 \le i \le N} |c_0 - y_i|$$

takes on its minimum. If one wants to phrase this differently, one could also say that we are looking for the optimal coefficient c_0^* for which

$$e(c_0^*) = \min_{c_0} \max_{1 \le i \le N} |c_0 - y_i|.$$

One could wonder if there is indeed only a single such value c_0, or if it may be possible to have a number of *different* values for c_0 for which the corresponding functions $p_0(x) = c_0$ are all best approximations to the data points.

Prove that the function $e(c_0)$ defined above has only a single minimum and that consequently there is exactly one, well-defined best l_∞ approximation among the constant functions. (Hint: Try your hands first on the case where there are only two data points, i.e. $e(c_0) = \max\{|c_0 - y_1|, |c_0 - y_2|\}$ and then generalize to N data points.)

Comment on what happens if you were looking for linear approximations $p_1(x) = c_0 + c_1 x$ with corresponding error function

$$e(c_0, c_1) = \max_{1 \le i \le N} |c_0 + c_1 x_i - y_i|.$$

Does this two-dimensional function have a single, unique minimum as well?

(4 points)