Homework assignment 9 – due Thursday 11/7/2013

Problem 1 (Best polynomial approximation). Compute, analytically (i.e. with exact values, not numerical floating point values), the best (least squares) polynomial approximation of degree 4 on the interval $[-1, 1]$ to the following functions:

a) $f(x) = \frac{1}{x}$;

a) $f(x) = e^x$.

Plot your best approximation $p_4(x)$ together with $f(x)$. 

Problem 2 (Least-squares approximation and other norms). In class, we defined the least-square approximating polynomial $p_q(x) = \sum_{k=0}^{q} a_k x^k$ as that polynomial that minimized the error

\[ e_2(a_0, \ldots, a_q) = \sum_{i=1}^{N} |p_q(x_i) - y_i|^2, \]

where we used the $l_2$ norm of the difference between $p_q(x_i)$ and $y_i$ (i.e., we squared the difference, and summed over it). We convinced ourselves that this then leads to a linear problem for finding the expansion coefficients $a_k$.

On the other hand, we could choose any other norm as well to define the error. For example, we could have chosen the $l_4$ norm

\[ e_4(a_0, \ldots, a_q) = \sum_{i=1}^{N} |p_q(x_i) - y_i|^4, \]

or the $l_\infty$ norm

\[ e_\infty(a_0, \ldots, a_q) = \max_{i=1}^{N} |p_q(x_i) - y_i|, \]

and finding the coefficients $a_k$ that minimize one of these functions will, most likely, yield a different polynomial $p_q(x)$ than the the one from before. Both of these polynomials are optimal, but with regard to different notions of the difference between $p_q(x_i)$ and $y_i$. It is relatively easy to see that if we choose any other norm than the $l_2$ norm, then the problem of finding the $a_k$ is a nonlinear one.

Take the same points from last week again:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_i$</td>
<td>1.51</td>
<td>2.01</td>
<td>2.49</td>
<td>2.98</td>
<td>3.51</td>
<td>4.01</td>
<td>4.49</td>
<td>5.02</td>
<td>5.52</td>
<td>5.98</td>
</tr>
</tbody>
</table>
Let us try to fit a linear function $p_1(x) = c_0 + c_1x$ to these points using the different notions of “error”. Do the following:

- For $r \in \{1, 2, 4, \infty\}$, plot the functions $e_r(c_0, c_1)$ as surfaces over the axes $c_0, c_1$. Determine whether these functions $e_r$ are smooth and whether, consequently, finding a minimum can be done by just setting the derivative to zero.

- For each of these values $r$, find the polynomial $p_q(x)$ that minimizes the error $e_r$. (For $r = 2$, this is the solution of Problem 4 of last week’s homework.) Plot all of these polynomials together in one plot in which you also show the 10 data points.

Repeat these computations for the following data set (the third to last data point has been changed: some large measurement error has occurred, or someone made a mistake transferring the device reading to the data sheet; or maybe this was what the experiment really gave):

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_i$</td>
<td>1.51</td>
<td>2.01</td>
<td>2.49</td>
<td>2.98</td>
<td>3.51</td>
<td>4.01</td>
<td>4.49</td>
<td>5.82</td>
<td>5.52</td>
<td>5.98</td>
</tr>
</tbody>
</table>

Comment on the suitability of the solutions you’ve found for approximating the two data sets.

**Note:** To compute each of these polynomials, you have to find the coefficients $c_0, c_1$ that minimize the respective error $e_r(c_0, c_1)$. In general, you will not be able to find these coefficients exactly except for the case $r = 2$. In this case, feel free to get approximate values of the coefficients by plotting $e_r$ as described above and visually determining values for which it is minimal. (8 points)