Problem 1 (Polynomial interpolation, again; last week’s homework didn’t show what I intended to show, see the answer sheets). Compute the polynomial \( p_{2N}(x) \) of order \( 2N \) such that
\[
\begin{align*}
&\bullet \quad p_{2N}(0) = 1, \\
&\bullet \quad p_{2N}(\pm \frac{1}{N}) = 0 \text{ for } j = 1, \ldots, N.
\end{align*}
\]
Plot these polynomials for \( N = 2, 4, 6, 8, 12, 20 \) in the interval \(-1 \leq x \leq 1\). What happens as \( N \) becomes larger? (3 points)

Problem 2 (\( L^\infty \) norm for functions). For vectors, the \( l^\infty \) norm equals the magnitude of the largest component of the vector. Similarly, for a function \( f(x), a \leq x \leq b \), we define the infinity norm (now written with an upper-case \( L^\infty \) to indicate that this is the norm of a function, rather than a vector) as
\[
\|f\|_{L^\infty(a,b)} = \max_{a \leq x \leq b} |f(x)|.
\]
Consider the functions \( p_{2N}(x) \) computed in Problem 1. These functions are made to interpolate data points \((x_i, y_i)\) for which all data points \(y_i\) lie in the range \(0 \leq y_i \leq 1\). Yet, as you should have seen from the graphs produced for Problem 1, \( p_{2N}(x) \) does not respect this range; the interpolating polynomials oscillate wildly between interpolation points.

For the 6 polynomials \( p_{2N}(x) \) computed in Problem 1 for \( N = 2, 4, 6, 8, 12, 20 \), compute \( \|p_{2N}\|_{L^\infty(-1,1)} \).
(Note: The maximum of a function \( f \) of course satisfies \( f' = 0 \). For the polynomial \( p_{2N} \) of order \( 2N \), this means that you are looking for a zero of a polynomial of order \( 2N - 1 \). This problem is not solvable in general if \( N \geq 3 \) – unless the coefficients of the polynomial satisfy some really lucky coincidence – so I will be satisfied if you can come up with any idea, rigorous or not, to find an approximate value of the infinity norm of \( p_{2N} \), as long as you explain how you compute it.) (3 points)

Problem 3 (Least-squares approximation). Take the same \( 2N + 1 \) points as in Problem 1. For \( N = 6, 8, 12, 20 \), compute the best least-squares approximating polynomials of order 4, i.e. the polynomial \( p_{4}(x) \) such that
\[
\left( \sum_{i=1}^{N} |p_{4}(x_i) - y_i|^2 \right)^{1/2}
\]
is minimal. Plot them for the range $-1 \leq x \leq 1$. Compare to the corresponding polynomials from Problem 1. What is the behavior of the least-squares approximates between the data points $(x_i, y_i)$?

(5 points)

Problem 4 (Extrapolation). We have measured the following 10 data points:

\[
\begin{array}{c|cccccccccc}
  x_i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
y_i & 1.51 & 2.01 & 2.49 & 2.98 & 3.51 & 4.01 & 4.49 & 5.02 & 5.52 & 5.98 \\
\end{array}
\]

It seems reasonable to assume a linear relationship between $x$ and $y$. Compute

- the interpolating polynomial $p^{\text{inter}}_9(x)$ for these 10 data points;
- the linear least-squares polynomial $p^{\text{ls}}_1(x)$ that best approximates these data points.

Plot both in the interval $-2 \leq x \leq 12$, together with the data points. If we want to extrapolate the measured behavior (i.e., predict the behavior of $y$ outside the range $1 \leq x \leq 10$ within which we have obtained measurements), what can you conclude from the plots? In particular, what are the values $p^{\text{inter}}_9(12)$ and $p^{\text{ls}}_1(12)$ that the two functions predict for $x = 12$? What value would you expect from the linear model that was clearly the basis on which the data points were obtained?  

(5 points)