MATH 601: Quiz 9 (11/07/2012)

Name: UIN:

Problem 1 (5 points): For each of the following mappings, state whether it is linear or not. If it is not linear, explain why.

- $F : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\quad F(v) := \begin{pmatrix} v_1 + 1 \\ v_2 + 1 \end{pmatrix} \quad$ where $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$.

  Answer: This mapping is not linear. We have shown in class that if F is linear then necessarily $F(0) = 0$. This condition is not true here, however: $F(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

- $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\quad F(v) := v - \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot v \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

  Answer: Yes, this mapping is linear. For example, let us consider the condition that requires that for every linear mapping $F(u + w) = F(u) + F(w)$:

  $$\begin{aligned} F(u+w) &= (u+w) - \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot (u+w) \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= u - \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot u \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} + w - \left(\begin{pmatrix} 1 \\ 2 \end{pmatrix} \cdot w \right) \begin{pmatrix} 2 \\ 1 \end{pmatrix} = F(u) + F(w). \end{aligned}$$

  The second condition is equally trivially verified.

- $F : P_2(t) \to \mathbb{R}^2$ defined by $\quad F(p(t)) := \begin{pmatrix} a \\ b \end{pmatrix} \quad$ when applied to a polynomial $p(t) = a + bt + ct^2 \in P_2(t)$.

  Answer: Yes, this mapping is also linear.

- $F : \mathbb{R}^{2\times 2} \to \mathbb{S}^{2\times 2}$ defined by $\quad F(A) := \frac{1}{2}(A + A^T) \quad$ when applied to a matrix $A \in \mathbb{R}^{2\times 2}$. ($\mathbb{S}^{2\times 2}$ is again the space of *symmetric* 2-by-2 matrices.)

  Answer: This one is linear as well.

- $F : \mathbb{R}^{2\times 2} \to \mathbb{R}$ defined by $\quad F(A) := \det(A) \quad$ when applied to a matrix $A \in \mathbb{R}^{2\times 2}$.

  Answer: This mapping is not linear. While $F(0) = 0$ (where the argument is the zero matrix), it is easy to verify from the definition of the determinant that $F(A + B) \neq \det(A) + \det(B)$. It is even simpler to verify that $F(kA) = k^2 A$ for $k \in \mathbb{R}$, also violating the requirements for a linear mapping.

(see backside)

Problem 2 (5 points): Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $F(v) := \begin{pmatrix} 1 & 3 \\ 2 & 6 \\ -3 & -9 \end{pmatrix} v$ where $v \in \mathbb{R}^2$. Answer the following questions about this mapping:

- Is it linear?

  Answer: Yes.

- What are its domain and image space?

  Answer: The domain is $\mathbb{R}^2$: one can apply this mapping to all two-dimensional vectors. The image space is $\mathbb{R}^3$ since the output of the mapping is a three-dimensional vector.

- What is the kernel, ker F, of this mapping and what is the dimension of ker F?

  Answer: The kernel of a mapping is the set of all vectors so that if the mapping is applied to such a vector the result is zero. In the current case, all vectors of the form $v = (3a, -a)^T$ for any $a \in \mathbb{R}$ are mapped to zero, so the kernel is the vector space of all vectors of this form:

  $$\ker F = \{v \in \mathbb{R}^2 : v \text{ has the form } v = (3a, -a)^T\}.$$

  The elements of this set form a line. The dimension of the kernel is one.

- What is the image, Im F, of this mapping and what is the dimension of Im F?

  Answer: The image of a mapping is the set of all possible results of the mapping. Given any input vector $v = (v_1, v_2)^T$ the output is always of the form $F(v) = (v_1 + 3v_2) \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. All of these points lie on a line. Consequently, the dimension of the image of F is also one.

- Is there an inverse of this mapping and if not why? An inverse mapping $B : \mathbb{R}^3 \to \mathbb{R}^2$ would satisfy $B(F(v)) = v$ for all $v \in \mathbb{R}^2$ and $F(B(u)) = u$ for all $u \in \mathbb{R}^3$.

  Answer: No, there is no inverse. As discussed in class, the dimension of the image of a mapping can not be larger than the dimension of its inputs. Since F maps $\mathbb{R}^2$ onto a single line in $\mathbb{R}^3$ (namely the line that forms its image), any mapping B from $\mathbb{R}^3$ to $\mathbb{R}^2$ would necessarily have to map this line into either a point or a line in $\mathbb{R}^2$. Thus, considering all points $v \in \mathbb{R}^2$, the points given by $B(F(u))$ would all lie along a single line in $\mathbb{R}^2$. In other words, there must be points u that do not lie on this line for which then consequently $B(F(u)) \neq u$.

  There are multiple other ways to think about this problem that all show that no such inverse can exist.