MATH 601: Quiz 7 (10/17/2012)

Name: UIN:

We call a set V a *vector space over the scalar field* K if

- a vector addition is defined so that $\forall u, v \in V : u + v \in V$
- a scalar multiplication is defined so that $\forall u \in V, k \in K : ku \in V$

and in addition these two operations satisfy the following eight properties ("axioms"):

A1: $\forall u, v, w \in V : (u + v) + w = u + (v + w)$

A2: $\exists w \in V$ so that $\forall u \in V$: $u + w = u$ (i.e., there is a unique zero element in the vector space)

A3: $\forall u \in V$ there exists $w \in V$ so that $u + w = 0$. We denote $w = -u$

A4: $\forall u, v \in V : u + v = v + u$.

M1: $\forall u, v \in V, k \in K : k(u + v) = ku + kv$

M2: $\forall u \in V, a, b \in K : (a + b)u = au + bu$

M3: $\forall u \in V, a, b \in K : (ab)u = a(bu)$

M4: With 1 being the unit element in K there holds $1u = u$.

In class, we have seen examples of vector spaces and examples of cases that are *not* vector spaces. For each of the following five examples, examine whether they are vector spaces; if you think that they are not, state at least one of the eight axioms that are violated. (2 points for each case.)

- $V = \mathbb{R}^2, K = \mathbb{R}$ where we define $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_2 \\ u_2 + v_1 \end{pmatrix}$, $k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$.

  Answer: Like in many of the other cases below, there are multiple axioms that are violated here. The simplest to verify is that with the addition defined here, $u + v \neq v + u$ (A4).

- $V = \mathbb{R}^2, K = \mathbb{C}$ where we define $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$, $k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$.

  Answer: Here, what happens is that if we multiply a vector with two elements in $\mathbb{R}$ by a number $k \in \mathbb{C}$ with a nonzero imaginary part, then we obtain a vector whose elements are no longer in $\mathbb{R}$. In other words, there are $k \in K$ so that $ku \notin V$ (violating the definition of a scalar multiplication).

- $V = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ (i.e., all vectors in $\mathbb{R}^2$ that are at most one unit away from the origin), $K = \mathbb{R}$ where we define $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$, $k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$.

  Answer: V contains only vectors with a distance from the origin less than or equal to one. On the other hand, because we use the usual definition of the vector addition, we can consider, for example $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ (both in V) and find that $u + v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \notin V$ (violating the definition of a vector addition).

- $V = \{x \in \mathbb{R}^2 : \|x\| \leq 1\}$ (i.e., all vectors in $\mathbb{R}^2$ that are at most one unit away from the origin), $K = \mathbb{R}$ where we define $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{\sqrt{(u_1+v_1)^2+(u_2+v_2)^2}} \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}$, $k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}$.

  Answer: Here, the addition has been modified in such a way that the sum of two vectors always has a distance from the origin equal to one (let's ignore for a moment the case where the fraction becomes singular). So the sum of two vectors is indeed again a vector in V. However, there no longer is a universal zero vector. For example, if we choose $w = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ as the zero vector then

  $$u + w = \frac{1}{\sqrt{u_1^2 + u_2^2}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

  which in general is not equal to u (A2).

- V is the set of all functions $f(t)$ where the argument t is from the interval $[0, 1]$ and where $f(t)$ satisfies $-1 \leq f(t) \leq 1$, $K = \mathbb{R}$. Addition and scalar multiplication of functions are defines as usual.

  Answer: This is essentially a variant of the third example: consider for example the function $f(t) = 1$ which is clearly in V. But $f + f$ is a function that is equal to 2 everywhere and so is no longer in V.