We call a set $V$ a vector space over the scalar field $K$ if

- a vector addition is defined so that $\forall u, v \in V : u + v \in V$
- a scalar multiplication is defined so that $\forall u \in V, k \in K : ku \in V$

and in addition these two operations satisfy the following eight properties (“axioms”):

\begin{align*}
A1: \quad & \forall u, v, w \in V : (u + v) + w = u + (v + w) \\
A2: \quad & \exists w \in V \text{ so that } \forall u \in V : u + w = u \text{ (i.e., there is a unique zero element in the vector space)} \\
A3: \quad & \forall u \in V \text{ there exists } w \in V \text{ so that } u + w = 0. \text{ We denote } w = -u \\
A4: \quad & \forall u, v \in V : u + v = v + u.
\end{align*}

\begin{align*}
M1: \quad & \forall u, v \in V, k \in K : k(u + v) = ku + kv \\
M2: \quad & \forall u \in V, a, b \in K : (a + b)u = au + bu \\
M3: \quad & \forall u \in V, a, b \in K : (ab)u = a(bu) \\
M4: \quad & \text{With } 1 \text{ being the unit element in } K \text{ there holds } 1u = u.
\end{align*}

In class, we have seen examples of vector spaces and examples of cases that are not vector spaces. For each of the following five examples, examine whether they are vector spaces; if you think that they are not, state at least one of the eight axioms that are violated. (2 points for each case.)

\begin{itemize}
  \item $V = \mathbb{R}^2, K = \mathbb{R}$ where we define
    \[
    \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}, \quad k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}.
    \]

  \textbf{Answer:} Like in many of the other cases below, there are multiple axioms that are violated here. The simplest to verify is that with the addition defined here, $u + v \neq v + u$ (A4).

  \item $V = \mathbb{R}^2, K = \mathbb{C}$ where we define
    \[
    \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \end{pmatrix}, \quad k \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} ku_1 \\ ku_2 \end{pmatrix}.
    \]

  \textbf{Answer:} Here, what happens is that if we multiply a vector with two elements in $\mathbb{R}$ by a number $k \in \mathbb{C}$ with a nonzero imaginary part, then we obtain a vector whose elements are no longer in $\mathbb{R}$. In other words, there are $k \in K$ so that $ku \notin V$ (violating the definition of a scalar multiplication).
\end{itemize}
• $V = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \}$ (i.e., all vectors in $\mathbb{R}^2$ that are at most one unit away from the origin), $K = \mathbb{R}$ where we define $(u_1, v_1) + (u_2, v_2) = (u_1 + v_1, u_2 + v_2)$, $k (u_1, v_1) = (ku_1, kv_1)$.

**Answer:** $V$ contains only vectors with a distance from the origin less than or equal to one. On the other hand, because we use the usual definition of the vector addition, we can consider, for example $u = \left( \begin{array}{c} 1 \\ 0 \end{array} \right), v = \left( \begin{array}{c} 0 \\ 1 \end{array} \right)$ (both in $V$) and find that $u + v = \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \notin V$ (violating the definition of a vector addition).

• $V = \{ x \in \mathbb{R}^2 : \|x\| \leq 1 \}$ (i.e., all vectors in $\mathbb{R}^2$ that are at most one unit away from the origin), $K = \mathbb{R}$ where we define $(u_1, v_1) + (u_2, v_2) = \frac{1}{\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2}} (u_1 + v_1, u_2 + v_2)$, $k (u_1, v_1) = (ku_1, kv_1)$.

**Answer:** Here, the addition has been modified in such a way that the sum of two vectors always has a distance from the origin equal to one (let’s ignore for a moment the case where the fraction becomes singular). So the sum of two vectors is indeed again a vector in $V$. However, there no longer is a universal zero vector. For example, if we choose $w = \left( \begin{array}{c} 0 \\ 0 \end{array} \right)$ as the zero vector then

$$u + w = \frac{1}{\sqrt{u_1^2 + u_2^2}} (u_1, u_2)$$

which in general is not equal to $u$ (A2).

• $V$ is the set of all functions $f(t)$ where the argument $t$ is from the interval $[0, 1]$ and where $f(t)$ satisfies $-1 \leq f(t) \leq 1$, $K = \mathbb{R}$. Addition and scalar multiplication of functions are defined as usual.

**Answer:** This is essentially a variant of the third example: consider for example the function $f(t) = 1$ which is clearly in $V$. But $f + f$ is a function that is equal to 2 everywhere and so is no longer in $V$. 
