MATH 601: QUIZ 6 (10/08/2012)

NAME: UIN:

Problem 1 (5 points): Consider the matrix $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Compute its inverse.

Answer: Starting with the linear system with multiple right hand sides,

$$AX = I$$

we can do the forward elimination and backward substitution to arrive at the linear system

$$IX = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}.$$

Since by definition, $X = A^{-1}$, we conclude that

$$A^{-1} = \begin{pmatrix} 3/4 & 1/2 & 1/4 \\ 1/2 & 1 & 1/2 \\ 1/4 & 1/2 & 3/4 \end{pmatrix}.$$

(see backside)

Problem 2: Let A be a square, invertible matrix and A^{-1} its inverse. We call a matrix A *symmetric* if $A = A^T$. Show that the statements below are true.
In the process of proving these, you may use the following: Recall that if A is a square, invertible matrix, then any matrix B that satisfies $BA = I$ (i.e., it is a left-inverse) also satisfies $AB = I$ (i.e., any left-inverse is also a right-inverse). The reverse statement is also true.

Part a (2 points): If A is symmetric, then the matrix $(A^{-1})^T$ is an inverse of A^T.

Answer: A^{-1} being an inverse of A implies that $A^{-1}A = I$. In other words, the matrices $A^{-1}A$ and I on the two sides of the equality sign are elementwise equal. Consequently, if we transpose both sides, they will still be identical, i.e., we know that $(A^{-1}A)^T = I^T$. Now, remember that first $(A^{-1}A)^T = A^T(A^{-1})^T$, and secondly that $I^T = I$. Consequently, we have just shown that $A^T(A^{-1})^T = I$. Given the statement above, this implies that $(A^{-1})^T$ is both a right-inverse and a left-inverse of A^T, i.e., an inverse of A^T.
This statement is actually always true, whether A is symmetric or not. If, furthermore, A is symmetric, i.e. $A^T = A$, then we have that $A(A^{-1})^T = I$, i.e., $(A^{-1})^T$ is also an inverse to A.

Part b (3 points): If A is symmetric, then the matrix A^{-1} is also symmetric.

Answer: From the previous statement we know that $(A^{-1})^T$ is an inverse to A. On the other hand, we know that A^{-1} is also an inverse. Since we know that the inverse of a matrix A is unique if A is invertible, we have just shown that $(A^{-1})^T = A^{-1}$ – i.e., that A^{-1} is symmetric.