MATH 601: Quiz 5 (10/03/2012)

Name: UIN:

Problem 1 (2 points): Consider the linear system $\begin{pmatrix} 1 & a \\ a & 9 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. For which values of the parameter a does the linear system have a unique solution? For which values of a (if any) does it have infinitely many solutions? For which values of a (if any) does it have no solutions?

Answer: To be invertible, the determinant of the matrix must be nonzero, i.e., we need to require that $9 - a^2 \neq 0$. This is the case for $a \neq \pm 3$. In other words, for $a \neq \pm 3$, the matrix is invertible and the solution of the problem is $x = \begin{pmatrix} 1 & a \\ a & 9 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. (And yes, we could compute the inverse of this matrix quite easily, but that is not material to the question at hand.)
The question what happens if $a = \pm 3$ is a bit more involved. In that case, the matrix is not invertible and the question if there are many or no solutions depends on the right hand side. To find out, we need to do the forward elimination step to get the matrix to triangular form. To this end, multiply the first equation by a and subtract it from the second. We then get: $\begin{pmatrix} 1 & a \\ 0 & 9 - a^2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 - a \end{pmatrix}$. For $a = +3$, the second equation now reads $0x_1 + 0x_2 = 2 - 3 = -1$. There are no possible values for x_1, x_2 so that this equation can be satisfied. For $a = -3$, the second equation reads $0x_1 + 0x_2 = 2 - (-3) = 5$ for which the same is true. In other words, if $a = \pm 3$, there are no solutions.

Problem 2 (4+4 points): For the following two linear systems, determine their unique solution. Alternatively, if there is no unique solution, state whether there are infinitely many solutions or no solutions at all. Also state, based on your answers to the previous questions whether the matrix is invertible.

Part a: $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 3 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

Answer: Using the usual steps to transform the system into triangular form (forward elimination) and then doing the backward substitution yields the solution $x = \begin{pmatrix} -2 \\ 6 \\ -3 \end{pmatrix}$.

Part b: $\begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 4 & 2 \\ 0 & 1 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$

Answer: This linear system has infinitely many solutions. After eliminating the first column, we have transformed the linear system to the following form:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

After another elimination step on the second column, we arrive at this:

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here, both the third and the fourth equations read $0 \cdot x = 0$ which any x satisfies. So we really have only two equations for four unknowns and that means that there are infinitely many solutions.

The problem I really wanted to pose would have had a right hand side equal to $(1, 2, 0, 2)^T$. In that case, after the two transformation steps, the third equation would have read $0 \cdot x = 0$ and the last one $0 \cdot x = 1$. If you only go by the third equation, then there are infinitely many solutions, but you have to take into account that there are no x so that the fourth equation is satisfied. As a consequence, the entire linear system has no solutions at all.