MATH 601: Quiz 1 (9/5/2012)

Name: UIN:

Problem 1: Show that for any two vectors $u, v \in \mathbb{R}^n$ there holds that

$$\text{proj}(u, v) = \text{proj}(u, \hat{v}),$$

i.e., that the projection of u onto the direction of v is the same as the projection of u onto the direction of the *direction of* v.

Verify that the equality above is indeed true for the following two vectors in $\mathbb{R}^2$: $u = (1, 2), v = (3, 4)$.

Answer: All you really have to remember is the formula for the projection, $\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v$ and that the direction of a vector is given by $\hat{v} = \frac{v}{\|v\|}$. Then compute as follows:

$$\text{proj}(u, \hat{v}) = \frac{u \cdot \hat{v}}{\|\hat{v}\|^2} \hat{v} = (u \cdot \hat{v})\hat{v} = \left(u \cdot \frac{v}{\|v\|}\right) \frac{v}{\|v\|} = \frac{u \cdot v}{\|v\|^2} v = \text{proj}(u, v),$$

where in the second equality we have used that $\|\hat{v}\| = 1$. This is easily seen: $\|\hat{v}\| = \left\| \frac{v}{\|v\|} \right\| = \frac{1}{\|v\|} \|v\| = 1$.

With the given two vectors, it is easy to verify that

$$\text{proj}(u, v) = \frac{u \cdot v}{\|v\|^2} v = \frac{11}{25}(3, 4)$$

and because $\hat{v} = \frac{1}{\sqrt{25}}(3, 4)$ we also have

$$\text{proj}(u, \hat{v}) = \frac{u \cdot \hat{v}}{\|\hat{v}\|^2} \hat{v} = \frac{\frac{11}{\sqrt{25}}}{1} \frac{1}{\sqrt{25}}(3, 4) = \frac{11}{25}(3, 4)$$

which is indeed the same as before.

Problem 2: In class we have defined the scalar product between two two-dimensional vectors $u, v \in \mathbb{R}^2$ as $u \cdot v = u_1 v_1 + u_2 v_2$. Let's assume that instead the scalar product was defined as

$$u \cdot v = u_1 v_1 + \tfrac{1}{2}(u_1 v_2 + u_2 v_1) + u_2 v_2.$$

Using this new scalar product, find two vectors u, v that are orthogonal. Sketch these two vectors and explain whether you would consider them to be orthogonal in the "usual" sense.

Answer: Since we are only asked to find *any* two vectors, let us choose $u = (1, 0), v = (1, v_2)$ and determine v_2 so that $u \cdot v = 0$, i.e., so that they are orthogonal with respect to this new scalar product: $0 = u \cdot v = 1 + \frac{1}{2} v_2$. This is satisfied by $v_2 = -2$. In other words, the two vectors $u = (1, 0), v = (1, -2)$ are orthogonal. If you plot them in a coordinate system, you will note that they are not orthogonal in the "usual" sense, though. In other words, the scalar product above is a valid mathematical definition, but it doesn't match our usual intuition.

Note that it is possible to choose vectors that happen to be both orthogonal in the "usual" sense as well as with regard to the new scalar product. This would have happened if we had chosen $u = (1, 1), v = (1, v_2)$ since we then obtain $v_2 = -1$. This is a correct answer to the problem but unfortunate because it doesn't quite show what was intended to show with this question.
