MATH 652: Optimization II

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Homework assignment 10 – due Thursday 4/22/2010

Problem 1 (PDE-constrained optimization). Geophysicists are interested in determining the structure of the earth interior by measuring how long seismic waves from earthquakes take to travel from the earthquake source to seismometer stations.

Let us consider the following, simplified situation: We have a one-dimensional medium that extends from $x = 0$ to $x = \infty$ (think, for example, of a semi-infinite string or rod). Waves are excited at the left end following a function $g(t)$ and travel at a (constant) speed of c to the right. The equations that describe these waves are

$$\begin{aligned}
\partial_t u(x,t) + c\partial_x u(x,t) &= 0 && \text{for } 0 < x < \infty, t > 0, \\
u(x,0) &= 0 && \text{for } 0 < x < \infty, t > 0, \\
u(0,t) &= g(t) && \text{for } t > 0.
\end{aligned}$$

These equations are called the *one-way wave equation*.

Let us say that at location $x = 1$ we have measured the signal for all times $0 \le t \le T$ to be $z(t)$ and that we would want to use this to determine the wave speed c. We could then pose the following PDE-constrained optimization problem:

$$\begin{aligned}
&\min_{u \in H^1, c \in \mathbb{R}} \frac{1}{2} \int_0^T (u(1,t) - z(t))^2 \, dt \\
&\text{subject to} \quad \partial_t u(x,t) + c\partial_x u(x,t) = 0 && \text{for } 0 < x < \infty, t > 0, \\
&\qquad\qquad\qquad\qquad u(x,0) = 0 && \text{for } 0 < x < \infty, t > 0, \\
&\qquad\qquad\qquad\qquad u(0,t) = g(t) && \text{for } t > 0.
\end{aligned}$$

Derive optimality conditions for this problem. Clearly indicate the function spaces from which each function (primal function, Lagrange multiplier, test function) comes. You may want to absorb the boundary condition at $x = 0$ into the function space as discussed in the example in class. Treat the time variable as we did for ODE problems, i.e. include the initial conditions in weak form. Note that all duality products are now integrals over both spatial and temporal variables. **(6 points)**

Problem 2 (Bonus problem). If you feel challenged, think (and write) a bit more about the previous problem in the following direction: For the one-way wave equation stated above, we can actually write down the solution – it consists of the signal at the left propagating unchanged to the right, yielding

$$u(x,t) = \begin{cases} g(t - \frac{x}{c}) & \text{for } t > \frac{x}{c}, \\ 0 & \text{otherwise.} \end{cases}$$

In other words, what we see at a point x at time t equals what we saw at $x = 0$ (i.e. the left boundary) but at the earlier time $t - \frac{x}{c}$. Note that $\frac{x}{c}$ is the time it took the signal to propagate from the left boundary to x at the wave speed c.

If we take this solution, then the optimization problem can also be written in the following, much simpler form:

$$\min_{c \in \mathbb{R}} f(c) = \frac{1}{2} \int_0^T \left[g\left(t - \frac{1}{c}\right) - z(t) \right]^2 dt.$$

If we are given a source function $g(t)$ and the measured signal $z(t)$, this is now simply a one-dimensional problem in the scalar wave speed c. You'd think this can't be overly complicated.

The problem is that earthquakes are oscillatory. Take, for example, a source of the following kind (plot it for yourself to see how it looks!):

$$g(t) = \begin{cases} \left(1 - \frac{(t-0.1)^2}{0.1^2}\right) \sin \frac{4\pi t}{0.1} & \text{for } 0 < t < 0.2, \\ 0 & \text{otherwise.} \end{cases}$$

Let us assume that we measure to time $T = 2$ and that the signal we get is

$$z(t) = \begin{cases} \left(1 - \frac{(t-1.1)^2}{0.1^2}\right) \sin \frac{4\pi (t-1)}{0.1} & \text{for } 1 < t < 1.2, \\ 0 & \text{otherwise.} \end{cases}$$

Write a program that (numerically or analytically) evaluates the objective function $f(c)$ for the range $0.5 \le c \le 2$ and plot it. How easy would you think it is to find its minimum? **(2 bonus points)**

In addition to this, continue to work on your semester project.

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!