MATH 652: Optimization II

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Partial answers for homework assignment 9

Problem 1 (An optimal control problem). In class, we had the example of a rover driving from x_A to x_B in a landscape in which the velocity attainable at position x given a force $q(t)$ was given by $\phi(x(t))q(t)$. The associated optimal control problem of finding the path of least energy consumption was given by

$$\min_{x(t)\in\mathbb{R}^2, q(t)\in\mathbb{R}^2} \frac{1}{2}\int_0^T q(t)^2 \mathrm{d}t$$
$$\dot{x}(t) = \phi(x(t))q(t),$$
$$x(0) = x_A,$$
$$x(T) = x_B.$$

Find the optimal control $q(t)$ and associated optimal path $x(t)$ for the following conditions:

$$x_A = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \qquad T = 1,$$
$$x_B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \phi = \begin{cases} 1 & \text{if } x_2 < 0, \\ 2 & \text{if } x_2 \geq 0. \end{cases}$$

Finding the solution to this problem is not easy if you don't know where to look. One way is to conjecture that the path will consist of two straight pieces from x_A to $(\alpha, 0)^T$ to x_B with some α, and that $q(t)$ is constant on each of these segments. Using this assumption, you can put a trial solution into the optimality conditions to see if you can satisfy all equations by using the open parameters (e.g. α and the magnitude of $q(t)$ on each segment). If you can satisfy all optimality conditions, you have found a solution (a local minimum), though there may of course be other solutions (local minima).

Answer. The Lagrangian for this problem reads

$$L(x(t), q(t), \lambda(t), \nu) = \frac{1}{2}\int_0^T q(t)^T q(t) \mathrm{d}t$$
$$- \int_0^T \lambda(t)^T (\dot{x}(t) - \phi(x(t))q(t)) \mathrm{d}t - \lambda(0)^T (x(0) - x_A) - \nu^T (x(T) - x_B).$$

From this, we can obtain the following optimality conditions:

$$\begin{aligned}
\dot{x}(t) &= \phi(x(t))q(t) \\
x(0) &= x_A \\
\dot{\lambda}(t) &= -\nabla\phi(x(t))[q(t)^T\lambda(t)] \\
\lambda(T) &= \nu \\
q(t) + \phi(x(t))\lambda(t) &= 0 \\
x(T) &= x_B.
\end{aligned}$$

Given the values for ϕ, x_A, x_B, T stated in the problem description, these optimality conditions are

$$\begin{aligned}
\dot{x}(t) &= (1 + H(x_2(t)))q(t) \\
x(0) &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
\dot{\lambda}(t) &= -\begin{pmatrix} 0 \\ \delta(x_2(t)) \end{pmatrix}[q(t)^T\lambda(t)] \\
\lambda(T) &= \nu \\
q(t) + (1 + H(x_2(t)))\lambda(t) &= 0 \\
x(1) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}$$

where

$$H(y) = \begin{cases} 0 & \text{if } y < 0, \\ 1 & \text{if } y \geq 0. \end{cases}$$

These equations are nonlinear, and we will have a hard time finding a solution if we don't already know where to look for. We will use the following strategy: we propose a trial solution that will contain a number of parameters; we will then plug it into the optimality conditions and see whether we can satisfy all equations by choosing the parameters appropriately.

Specifically, let us assume that the optimal path consists of two parts, one that leads along a straight line from x_A to an intermediate point $x_i = (\alpha, 0)^T$, using a constant force q on time interval $[0, t_i)$. Then, in the time interval $(t_i, 1]$ we move along a straight line with constant speed from x_i to x_B.

Let us consider the first part. Since the path goes entirely in the lower half plane, the optimality conditions on this part read

$$\begin{aligned}
\dot{x}(t) &= q(t) \\
x(0) &= \begin{pmatrix} -1 \\ -1 \end{pmatrix} \\
\dot{\lambda}(t) &= 0 \\
q(t) + \lambda(t) &= 0 \\
x(t_i) &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix}.
\end{aligned}$$

The first and last equation, together with the assumption that $q(t)$ is constant on each part of the path, yield $\dot{x}(t) = q(t) = \frac{1}{t_i}(\alpha+1, 1)^T$. On this first part, we then also have $\lambda(t) = -\frac{1}{t_i}(\alpha+1, 1)^T$.

We can do the same argument for the second part of the path. The optimality conditions there are

$$\begin{aligned}
\dot{x}(t) &= 2q(t) \\
x(t_i) &= \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \\
\dot{\lambda}(t) &= 0 \\
\lambda(T) &= \nu \\
q(t) + 2\lambda(t) &= 0 \\
x(1) &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\end{aligned}$$

If the path is straight and at constant velocity, we will find again from the first, second, and last conditions that $\dot{x}(t) = 2q(t) = \frac{1}{1-t_i}(1-\alpha, 1)^T$. Likewise, the second to last condition implies $\lambda(t) = -\frac{1}{2(1-t_i)}(1-\alpha, 1)^T$.

We now have two parameters, α, t_i that we need to determine. We can use the jump conditions in the original optimality conditions,

$$\dot{\lambda}(t) = -\begin{pmatrix} 0 \\ \delta(x_2(t)) \end{pmatrix} [q(t)^T \lambda(t)]$$

for this purpose. This equation has $x-$ and $y-$components. Let us consider the $x-$component first: $\dot{\lambda}_1 = 0$. We have previously established that for $t < t_i$, $\lambda_1(t) = -\frac{1}{t_i}(\alpha+1)$ and for $t > t_i$, $\lambda_1(t) = -\frac{1}{2(1-t_i)}(1-\alpha)$. Since the time derivative must be zero across this interface, we see that $-\frac{1}{t_i}(\alpha+1) = -\frac{1}{2(1-t_i)}(1-\alpha)$, or equivalently:

$$(2-2t_i)(\alpha+1) = t_i(1-\alpha). \tag{1}$$

The $y-$component of the condition reads as follows:

$$\dot{\lambda}_2 = -\delta(x_2(t))[q(t)^T \lambda(t)].$$

To make sense of this, let us integrate this equation from $t_i - \epsilon$ to $t_i + \epsilon$:

$$\lambda_2(t_i + \epsilon) - \lambda_2(t_i - \epsilon) = -\int_{t_i-\epsilon}^{t_i+\epsilon} \delta(x_2(t))[q(t)^T \lambda(t)] \,\mathrm{d}t.$$

Before and after t_i, the term $q(t)^T\lambda(t)$ is constant, but it could be discontinuous at t_i. Let us call the respective values $f^{\pm}$, then using what we've found out about $q(t)$ and $\lambda(t)$ above, we see that

$$\begin{aligned} f^- &= q(t)^T\lambda(t)|_{t=t_i-} = -\frac{1}{t_i^2}[(\alpha+1)^2+1], \\ f^+ &= q(t)^T\lambda(t)|_{t=t_i+} = \frac{1}{2}\frac{1}{(1-t_i)^2}[(1-\alpha)^2+1]. \end{aligned}$$

Then, with $f(t-t_i)$ the function that takes on f^- or f^+ for $t - t_i < 0$ and $t - t_i > 0$, respectively, we have

$$\lambda_2(t_i + \epsilon) - \lambda_2(t_i - \epsilon) = -\int_{t_i-\epsilon}^{t_i+\epsilon} \delta(x_2(t)) f(t-t_i) \,\mathrm{d}t.$$

Let us transform the integration variable from t to x_2. We then get

$$\lambda_2(t_i + \epsilon) - \lambda_2(t_i - \epsilon) = -\int_{-\epsilon}^{\epsilon} \delta(x_2) \frac{f(x_2)}{\dot{x}_2} \,\mathrm{d}x_2.$$

We have here made use of the fact that $f(x_2) = f^{\pm} = f(t-t_i)$ because $x_2(t)$ switches from negative to positive at the same time as $t - t_i$ switches from negative to positive.

Now, to evaluate the integral, think of the delta function as a Gaussian function that we make narrower and narrower. Since the rest of the integrand, $\frac{f(x_2)}{\dot{x}_2}$ is piecewise constant, we get

$$\lambda_2(t_i + \epsilon) - \lambda_2(t_i - \epsilon) = -\frac{1}{2}\left[\frac{f^+}{\dot{x}_2^+} + \frac{f^-}{\dot{x}_2^-}\right] = -\frac{1}{2}\left[-\frac{q^Tq|_{t_i+}}{4q_2|_{t_i+}} - \frac{q^Tq|_{t_i-}}{q_2|_{t_i-}}\right]$$

On the other hand, we know the left hand side too:

$$-\frac{1}{2(1-t_i)} + \frac{1}{t_i} = -\frac{1}{2}\left[-\frac{q^Tq|_{t_i+}}{4q_2|_{t_i+}} - \frac{q^Tq|_{t_i-}}{q_2|_{t_i-}}\right]. \tag{2}$$

The equations (1) and (2) now give us two conditions for the two variables t_i, α. There is no simple closed form solution to this problem, but we can numerically solve it to find

$$t_i = 0.54559..., \qquad \alpha = -0.53826...$$

Not coincidentally, using these values, we can see that $\|q(t)|_{t=t_i-}\| = \|q(t)|_{t=t_i+}\|$, i.e. the magnitude of the force applied is constant throughout the path – just the speed is different.