MATH 652: Optimization II

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Partial answers for homework assignment 1

Problem 1 (l_∞ minimization). Assume you are given the following time series:

t_i	0	1	2	3
y_i	1.1	1.9	2.8	3.2

Consider the problem of fitting a line $y(t) = at + b$ through this data set. One way to do so is to ask for that set of parameters $x = \{a, b\}$ for which the maximal deviation $f(x) = \max_{i=1...4} |y_i - y(t_i)|$ is minimal. Note that the right hand side depends on x through the equation for $y(t)$.

Re-state this problem as a linear programming problem by introducing a (single) slack variable s. This way, you get a linear optimization problem in three variables: a, b, s.

While we will leave finding the solution of such problems for later, try to visualize the feasible set of this problem, i.e. the set of all points $\{a, b, s\}$ that satisfy the constraints of the re-formulated problem.

Answer. We can reformulate the problem

$$\min_{x=\{a,b\}} f(x)$$

where $f(x) = \max_{i=1...4} |y_i - y(t_i)|$ by introducing a "slack" variable s. By requiring that $s \geq |y_i - y(t_i)|$ for $i = 1, 2, 3, 4$, we know that s is greater than or equal to the maximum of all the residuals $|y_i - y(t_i)|$. A first reformulation would then be this:

$$\min_{\{a,b,s\}} s$$
$$s \geq |y_i - y(t_i)| \qquad \forall i = 1, \dots, 4.$$

Note that if we make s as small as possible, we will certainly make it so small that it equals at least one of the residuals (namely the largest one of the residuals).

Unfortunately, the problem is still nonlinear (and also not differentiable). We can avoid this by using the following variant:

$$\min_{\{a,b,s\}} s$$
$$s \geq y_i - y(t_i) \qquad \forall i = 1, \dots, 4,$$
$$s \geq -(y_i - y(t_i)) \qquad \forall i = 1, \dots, 4.$$

The reason why this works is because if s is greater than the difference $y_i - y(t_i)$ and also greater than its negative, then it must be greater than the greater of the two of them, i.e. of the magnitude of the residual! This problem is now linear. We can slightly reformulate it to get it into a form that we are more familiar with:

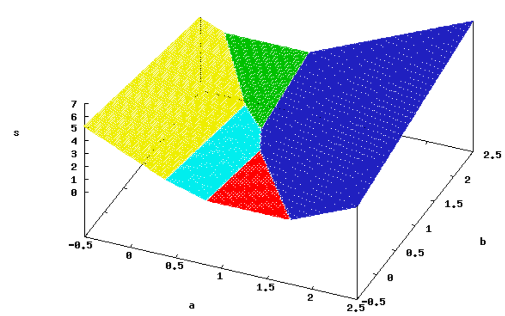
$$\begin{aligned}
&\min_{\{a,b,s\}} s \\
&\qquad s + t_i a + b \geq y_i \qquad\qquad \forall i = 1, \ldots, 4, \\
&\qquad s - t_i a - b \geq -y_i \qquad\qquad \forall i = 1, \ldots, 4.
\end{aligned}$$

This, in turn can be written as follows, using the values for t_i, y_i:

$$\min_{x=\{a,b,s\}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \begin{pmatrix} a \\ b \\ s \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 \\ -0 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 1 \\ 2 & 1 & 1 \\ -2 & -1 & 1 \\ 3 & 1 & 1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ s \end{pmatrix} \geq \begin{pmatrix} 1.1 \\ -1.1 \\ 1.9 \\ -1.9 \\ 2.8 \\ -2.8 \\ 3.2 \\ -3.2 \end{pmatrix}.$$

The second part of the problem asked to depict the feasible set of this problem. It is the volume above the surface shown here:



Note that the feasible set is unbounded from above. That this so is clear by considering that for the formulation we had above, only constraints $s \geq \ldots$ existed. In other words, nothing disallows s from being as large as it wants. On the other hand, the lack of such constraints is not of any harm: The objective function wants to find the smallest possible value of s.