MATH 442: Mathematical Modeling

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Group project – due 10/28/2010

Each group should jointly work on the following problem. Your task is to work on the various parts, implement the necessary methods and produce a write-up that describes your results. Your report should include an introduction in which you state the problem including all formulas, a methods section in which you discuss how you approached the problem and how you implemented it, and a results section in which you cover your results comprehensively. Do not just state what you find but also try to interpret the results. Your report should be around 10 pages, including figures. Write this report in LyX. You also need to provide your program code in electronic form to me. Programs may be written in Maple or any other programming environment your group feels comfortable with.

The solar system

In this project, we will investigate the motion of the bodies in our solar system. We know that planets move around the sun in an orderly fashion: by and large, everyone stays on her orbit for long times without interfering with other planets too much. But why is this so? This project is about investigating some of the conditions under which the solar system would not be as stable as it looks to us.

In previous homework you were asked to derive a model for how multiple bodies in a solar system interact with each other gravitationally. Let's assume that there is a single central star (which we denote by an index one, for convention) and $N-1$ other bodies in this system, with masses $m_1, \ldots, m_N$, respectively. Let their three-dimensional positions at time t be $\mathbf{r}_i(t), i = 1 \ldots N$, then Newton's laws tell us that they move according to the equations

$$m_i \frac{d^2}{dt^2}\mathbf{r}_i(t) = \mathbf{F}_{i,\text{total}}(t),$$

where

$$\mathbf{F}_{i,\text{total}}(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} -G\frac{m_i m_j}{|\mathbf{r}_{ij}(t)|^3}\mathbf{r}_{ij}(t)$$

$$\mathbf{r}_{ij}(t) = \mathbf{r}_i(t) - \mathbf{r}_j(t).$$

To make this model complete, we need initial positions $\mathbf{r}_i(0)$ and initial velocities $\mathbf{v}_i(0) = \frac{d}{dt}\mathbf{r}_i(0)$.

Part 1. For a time of your choice (what we'll call the *initial time*), find out the position and velocity of the eight planets as well as Pluto and the five largest asteroids Ceres, Vesta, Pallas, Hygiea and Interamnia. Start with the Wikipedia pages on the planets and asteroids, and Wikipedia pages they refer to; at the bottom of each page there is a list of external links from which you should find your way to various websites that list the *proper orbital elements* of each of these bodies. You may assume that at your initial time the sun is at the origin of the coordinate system.

For the formulas above you also need the masses of each of these 15 bodies, which are readily available from Wikipedia. Provide all this data you find as part of your report.

Note: This part seems like it should be a quick find but it may in fact take quite a while to have all this information available and converted into the necessary three-dimensional coordinates for initial position and velocity. Do not leave this part for the last minute!

Part 2. Use Maple (or any other programming system your group may agree upon) to numerically solve the differential equations above for the system of 15 objects for at least 10,000 years and if you can for the longest time for which you can let your simulation run. Plot the trajectories of the planets over this time horizon and verify that they move on stable orbits that move at best by very small amounts. You may have to play with the kind of numeric solver and accuracy in dsolve to achieve this.

Part 3. The Borg (of Star Trek fame) are growing old and get ready to settle down. For reasons unknown to us they find Jupiter a friendly place. However, they still have a glimmer of evil in them so the spaceship with which they arive slows down Jupiter by a factor of ten before it drops off its colonists and gets the heck out of here. Re-run your simulation where Jupiter's initial velocity is only one tenth of what it was in Part 1 and observe the effect on the solar system. What happens? Plot again the trajectories of all solar system bodies over your time horizon of at least 10,000 years. Interpret your findings.

Note: In addition to a plot that shows the entire trajectories at once, try to also generate a plot that zooms in to only show the fraction of the universe close to the sun, maybe the innermost 10 billion kilometers or even smaller. This should provide a more accurate intuition of what is going on here.

Part 4. In a bout of over-colonization, humanity had also played with the stability of the solar system: They found themselves a nice habitable Class M planet in the form of Mars, but it needed to get closer to Sun to become cozy and warm. Consequently, Mars was towed towards Sun and dropped off (at the initial time of your simulation) at a point exactly 1.2 times as far away from Sun as Earth along the line Sun-Earth. However, due to a last-minute tractor beam malfunction, its initial velocity – while perpendicular to the line Sun-Earth-Mars – was only one tenth that of Earth at the initial time.

Compared to the simulations made in Part 2, all that has changed now is the initial position and velocity of Mars, in much the same way as what changed in Part 3. Simulate this situation as well and determine what happens in the long run to the solar system. Compare the outcome to that of Part 3.

Part 5. Another way to disturb the orbits of the planets is to assume that there was a second big object in the solar system. While Jupiter is big (318 times as massive as Earth), it is still fairly small compared to the sun: it has less than 1,000th of the Sun's mass. Repeat the computations from Part 2 under the assumption that Jupiter had half the mass of the sun. At this mass Jupiter would be a star in itself, making the solar system a binary system with a bunch of planets thrown in. Following your computations, what can you say about the stability of planetary orbits in binary systems?

Bonus part (worth an additional 5% of the project). If you feel challenged, consider the interaction of stars in a dense star cluster like Messier 92 (look it up on Wikipedia!). To this end populate your simulation with not just 15 objects, but with 1,000 or 10,000 stars with randomly chosen locations in a volume representative of Messier 92. Their masses should equal one tenth to 1000 times the mass of the Sun, again randomly chosen. Equip each star with a velocity that appears reasonable (for example a velocity that allows stars to cross the star cluster in 100–1000 years). Plot trajectories of all your stars for a representative time horizon and interpret what you find.

Caution: If enough stars are involved, compute times can become fairly long. Consequently, try this with only a handful of stars at first and then increase their number as you become comfortable with the correctness of results.

I try to be as good a teacher as possible, but to succeed in this goal I need feedback from those who see me teach, i.e. you. If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, group work vs. whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!