MATH 442: Mathematical Modeling

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Homework assignment 5 – due 10/7/2010

Problem 1 (Some basic calculus in one variable). Given the function of one argument

$$f(x) = e^{3x}$$

derive by hand the Taylor series up to order 3 around the point $\bar{x} = 0$.

(2 points)

Use Maple to find the Taylor expansion of the function

$$f(x) = \ln(x)e^{x}$$

around the point $\bar{x} = 1$ up to order 8. Provide a single plot in which you show $f(x)$ and the Taylor expansion from order 0 to 8, i.e. a total of 10 curves.

(3 points)

Problem 2 (Some basic calculus in several variables). Given the function of two argument

$$f(x_1, x_2) = e^{x_1} \sin x_2$$

derive by hand the Taylor expansion up to order 1 (i.e. the constant and linear terms) around the point $\bar{x} = (0, 0)$. **(2 points)**

Use Maple to find the Taylor expansion of the function above around the point $\bar{x} = (1, \pi)$ up to order 2 (i.e. including all constant, linear, and quadratic terms). Provide a single plot in which you show $f(x)$ and the quadratic Taylor expansion. **(3 points)**

Problem 3 (Dimensional analysis). Newton's equations state that gravity acts on two bodies of masses m_1, m_2 at a distance r through an attracting force of strength

$$F = G\frac{m_1 m_2}{r^2},$$

where $G = 6.67428 \cdot 10^{-11} \text{m}^3\text{kg}^{-1}\text{s}^{-2}$. Earth has a mass of $m_1 = 5.9737 \cdot 10^{24}$ kg and the sun has a mass of $m_2 = 1.98892 \cdot 10^{30}$ kg. The mean distance between earth and the sun is $r = 1.49598261 \cdot 10^{11}$ m. On the other hand, Newton's law says that the acceleration a to which earth is subject satisfies $F = m_1 a$. In other words, we have that

$$a = G\frac{m_2}{r^2}.$$

Use the constants that appear in this second relation to define a "typical time" for the gravitationally bound system earth-sun. Can you somehow relate this "typical" time scale to a time scale you are familiar with? (Hint: r is the linear distance between the two objects, but there may be a constant related to a circle involved). **(4 points)**

Problem 4 (Solar systems and similar). Solar systems (and galaxies, and star clusters, etc) consist of a number of bodies all of which act gravitationally on each other. Consider, for example, the solar system as it was known in classical time: it had the sun and the six major planets Mercury, Venus, Earth, Mars, Jupiter and Saturn that can be seen without a telescope. Let us start with the following statements:

- The positions of the sun at time t is given by the three-dimensional vector $\mathbf{x}_0(t)$.
- The positions of the six planets at time t is given by the three-dimensional vectors $\mathbf{x}_i(t)$ where $i = 1$ for Mercury and $i = 6$ for Saturn.
- Sun has mass m_0, the planets have masses $m_1 \ldots m_6$.
- Newton's laws of gravitation describe the motion of each of the bodies in the solar system, i.e.

  $$\begin{aligned} m_0 \mathbf{a}_0(t) &= \mathbf{F}_{0,\text{total}}(t), \\ m_i \mathbf{a}_i(t) &= \mathbf{F}_{i,\text{total}}(t), \qquad i = 1 \ldots 6. \end{aligned}$$

  Here, $\mathbf{a}(t) = \frac{d^2}{dt^2}\mathbf{x}(t)$ is the acceleration of a body, i.e. the time derivative of its velocity or equivalently the second time derivative of its position. The right hand side is the sum of all forces acting on a body.

- Body p ($k = 0 \dots 6$) in the solar system acts gravitationally on body q ($q = 0 \dots 6$ but $q \neq p$) by pulling on body q with a force equal to

$$\mathbf{F}_{pq} = -G \frac{m_p m_q}{|\mathbf{r}_{pq}|^3} \mathbf{r}_{pq},$$

  where G is the gravitational constant, m_p, m_q are the respective masses of the two bodies, and $\mathbf{r}_{pq}$ is the vector that points from p's position to q's position.

Answer the following questions:

- Express $\mathbf{r}_{pq}$ in terms of the positions of the bodies in the solar system. **(2 points)**
- What is the total force acting on each of the bodies? **(2 points)**
- What is the differential equation that describes each body's motion? How many equations do you have in total? **(4 points)**
- How many initial conditions do you need for each body? How many in total? What kind of initial conditions would one prescribe, i.e. what is their physical meaning? **(4 points)**

I try to be as good a teacher as possible, but to succeed in this goal I need feedback from those who see me teach, i.e. you. If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, group work vs. whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!