# MATH 651: Optimization I

Lecturer: Prof. Wolfgang Bangerth
Blocker Bldg., Room 507D
(979) 845 6393
bangerth@math.tamu.edu
http://www.math.tamu.edu/~bangerth

## Homework assignment 10 – due Thursday 12/03/2009

**Problem 1 (Optimality conditions, part 1)** Consider the following problem:

$$\min_{x\in\mathbb{R}^2} \quad x_1^2 + x_2$$
$$x_2 \geq 0.$$

Answer the following questions: (i) What is the solution? (ii) What are the necessary first order conditions for this problem? (iii) Verify that the solution indeed satisfies the first order conditions. (iv) What are the second order sufficient conditions for this problem? (v) Verify that the solution indeed satisfies the second order sufficient conditions. **(4 points)**

**Problem 2 (Optimality conditions, part 2)** Repeat the same steps as in problem 1 for the following version of the problem:

$$\min_{x\in\mathbb{R}^2} \quad x_1^2 + x_2^2. \qquad\qquad x_2 \geq 0.$$

What is the additional complexity with this problem? **(4 points)**

**Problem 3 (Optimality conditions, part 3)** Consider the following problem:

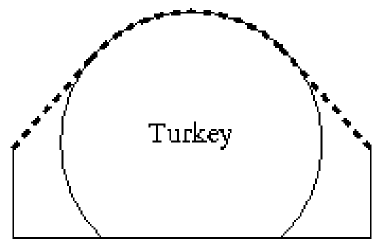
$$\min_{x\in\mathbb{R}^2} \quad -\|x\|_{\ell_2}^2,$$
$$\|x\|_{\ell_\infty} \leq 1.$$

The objective function is not convex (in fact, it is concave) and the constraint is not differentiable. However, the problem can be reformulated as follows:

$$\min_{x\in\mathbb{R}^2} \quad -\|x\|_{\ell_2}^2,$$
$$-x_1 + 1 \geq 0, \qquad x_1 + 1 \geq 0, \qquad -x_2 + 1 \geq 0, \qquad x_2 + 1 \geq 0.$$

Find all four (local) solutions of this problem. Verify that at one of these points the first and second order necessary conditions are satisfied. **(4 points)**

**Problem 4 (After Thanksgiving).** The turkey has been roasted, some has been eaten, it was too much as always, and so the rest of the bird has to go into the fridge. Since turkeys have a tendency to be dry, the family wisdom calls for a protective layer of cellophane (cling wrap). To make our lives somewhat easier, let's assume pan, turkey (circle), and wrap (dashed line) live in a two-dimensional world:




The pan is 50cm wide, its rim 10cm high, and the turkey has a radius of 15cm is centered left-right in the pan and its center is 10cm above the bottom of the pan.

We want to compute the shape of the cling wrap. To this end, assume we can describe it as a sequence of line segments between $N+1$ points $(x_i, y_i)$ where $x_i = \frac{i}{N}50cm, i = 0, \dots, N$ are given points. Then the energy needed to stretch the cellophane into the shape described by the values $y_i$ is

$$f(y) = \sum_{i=1}^{N} \frac{1}{2} \left( \frac{y_i - y_{i-1}}{x_i - x_{i-1}} \right)^2 + \sum_{i=0}^{N} (y_i - 10cm) \tag{1}$$

Nature typically adjusts things in such a way that the energy is minimized, i.e. the shape of the plastic will be so that $f(y)$ is minimal. Do the following: (i) Derive the standard form of an inequality constrained problem for minimizing $f(y)$ subject to the constraint that at $x_0, x_N$ the cellophane is at $y_0 = y_N = 10cm$ and that at each of the points inbetween the cellophane is above or on the turkey and above or on the pan. (ii) Derive the first order necessary conditions any local optimum has to satisfy under the assumption that the LICQ holds. (iii) Using your method of choice, find the minimum of this optimization problem for $N = 20$. (iv) Plot the solution and state the minimal energy. **(6 points)**

This is a little trickier: State the form of the LICQ for your constraints and verify that at the solution the LICQ will hold. **(2 bonus points)**

If you can, repeat steps (iii) and (iv) for $N = 1000$. **(2 bonus points)**

*If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!*