MATH 651: Optimization I

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Homework assignment 3 – due Thursday 10/1/2009

Problem 1 (Search directions). The steepest descent method chooses a search direction $\delta x_k = -\nabla f(x_k)$. By Taylor's theorem it is easy to show that this direction is a direction of descent, i.e.: there exists a number T so that if we go in direction δx_k no more than T times $\|\delta x_k\|$, our objective function will become smaller (we hope of course that eventually we will land in a minimum). In formulas: for this choice of δx_k, there exists $T > 0$ so that

$$f(x_k + \alpha \delta x_k) < f(x_k) \qquad \forall 0 < \alpha < T.$$

Prove that a similar statement (possibly with a different value for T) is also true if we choose as search direction $\delta x_k = -B\nabla f(x_k)$ for any given matrix B that is positive definite. **(3 points)**

Problem 2 (Radius of convergence for Newton's method). You have seen in class that Newton's method (with step length $\alpha = 1$) can only be proven to converge if we have a starting point x_0 that is close enough to the solution x^*. One reason may be that far enough away from the minimum x^* the function $f(x)$ may not be convex any more. Explain in words for a function $f(x)$ of a single variable $x \in \mathbb{R}^1$ what would go wrong if we started in an area where $f(x)$ is not convex (i.e. where $f''(x) < 0$). In particular, think about how Newton's method chose its search direction. If you want a concrete example to explain things with, use the function

$$f(x) = -\frac{1}{1+x^2}$$

whose minimum is at $x^* = 0$ but which is positive definite only in an interval around the origin. You can illustrate your explanation with numerical results that show what Newton's method does if, for example you start at $x = 0.1, 0.5, 1, 2, 5, \ldots$. **(4 points)**

Problem 3 (Radius of convergence for Newton's method). The case discussed in the previous problem is not the only one where Newton's method may not converge. Consider

$$f(x) = x \ \arctan x - \frac{1}{2}\ln(1+x^2).$$

This function is convex everywhere since $f''(x) = \frac{1}{1+x^2} > 0$. Yet, Newton's method (with step length $\alpha = 1$) only converges if started within an interval $[-r, r]$ around the minimum $x^* = 0$. Determine numerically the radius of convergence r for this problem. What happens if you start with $x_0 = r$? What if $x_0 > r$? **(5 points)**

Problem 4 (Rate of convergence for Newton's method). For the same function as in the previous problem, verify numerically that whenever Newton's method converges with step length $\alpha = 1$, that it converges with quadratic order. To do so start at a point x_0 within $[-r, r]$, calculate x_k for a number of iterations k, and test that the sequence $x_k - x^*$ indeed converges to zero with quadratic order. **(3 points)**

Problem 5 (Newton's method with line search). For the same function as in the previous problems, implement Newton's method with a line search procedure to determine the step length α_k in each step, i.e. we no longer assume that we can work with $\alpha = 1$. Using your implementation, show the following: (i) the algorithm now converges for all starting points x_0, even those that lie far outside the interval $[-r, r]$; and (ii) when close to the solution, the line search criterion allows a step length of $\alpha_k = 1$ to be chosen, guaranteeing quadratic convergence once we get close to the solution. **(4 points)**

If you have comments on the way I teach – in particular suggestions how I can do things better, if I should do more or less examples, powerpoint slides vs whiteboard, etc – or on other things you would like to critique, feel free to hand those in with your homework as well. I want to make this as good a class as possible, and all comments are certainly much appreciated!