MATH 417: Numerical Analysis

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Homework assignment 9 – due 4/19/2007

Problem 1 (Lagrange interpolation, repeated). The polynomial $p_4(x)$ calculated in Problem 3 of last week's homework by construction interpolates the function $f(x) = \log x$. Compute an upper bound for the error on the interval $[1, 2]$, using the theorem that states how large $|f(x) - p_4(x)|$ can at most be.

(3 points)

Problem 2 (Lagrange interpolation). For the data set $x_i = \{1, 2, 3, 4, 5\}$, $y_i = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\}$, compute the Lagrange interpolation polynomial. Plot this polynomial in the interval $-5 \leq x \leq 10$ together with the function $f(x) = \frac{1}{x}$ and describe, in words, where the interpolating polynomial is a reasonable approximation of $f(x)$.

(3 points)

Problem 3 (Lagrange interpolation of higher order). For each of the values $N = 1, 2, 4, 6, 8, 12, 20$, compute the polynomial $p_{2N}(x)$ of order $2N$ such that

- $p_{2N}(0) = 1$,
- $p_{2N}(\pm\frac{j}{N}) = 0$ for $j = 1, \dots, N$.

Plot these polynomials in the interval $-1 \leq x \leq 1$ (for better visibility, restrict the y-range to $-10 \dots 10$). What happens as N becomes larger? (Hint: You will want to compute the polynomials with a computer algebra system or a self-written program, since computing polynomials of degree 40 on paper becomes tedious. You can make your life a lot easier by only computing those polynomials that you actually need.)

(6 points)

Problem 4 (Non-equidistant Lagrange interpolation). Modify your program for Problem 3 to solve the interpolation problem

- $p_{2N}(0) = 1$,
- $p_{2N}\left(\sin\left(\pm\frac{\pi j}{2N}\right)\right) = 0$ for $j = 1, \dots, N$

for all values of N in problem 2. Note that the interpolation points $\sin\left(\pm\frac{\pi j}{2N}\right)$ are between -1 and 1 as before, but are now no longer equidistantly spaced.

(3 points)

Problem 5 (Numerical differentiation). In class, the symmetric second difference quotient

$$f''(x) \approx \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}$$

was introduced. Here, we want to study its properties.

(a) Compute the quadratic Lagrange interpolation polynomial $p_2(x)$ that interpolates f in the points $x-h$, x and $x+h$ and show that the formula is the second derivative $p_2''(x_0)$ of this polynomial.

(b) Show that the formula is exact for all polynomials of degree at most 3 (Hint: show this for the monomials $x^k, k = 0, 1, 2, 3$ and explain why this is sufficient).

(c) Use the Taylor polynomial of degree 3 for f around the point x and its remainder term to show that

$$f''(x) - \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = -\frac{h^2}{12} f^{(4)}(\xi)$$

for some $\xi \in (x-h, x+h)$. **(6 points)**

Problem 6 (Finite difference approximation of the derivative). Take the function defined by

$$f(x) = \begin{cases} \frac{1}{2}x^3 + x^2 & \text{for } x < 0 \\ x^3 & \text{for } x \geq 0. \end{cases}$$

Compute a finite difference approximation to $f'(x_0)$ at $x_0 = 1$ with both the one-sided and the symmetric two-sided formula. Use step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{64}$. Determine experimentally the convergence orders you observe as $h \to 0$.

Repeat these computations for $x_0 = 0$. What convergence orders do you observe? Why? **(4 points)**

Problem 7 (Derivatives of an implicit function). Let $f(x)$ be defined implicitly as follows: for every $x > 0$, $f(x)$ is that value y for which

$$ye^y = x. \tag{1}$$

In other words, every time one wants to evaluate $f(x)$ for a particular value x, one has to solve equation (1) for y. This can be done using Newton's method, for example, or any of the other root finding algorithms we had in class applied to the function $g(y) = ye^y - x$. As a sidenote, the function $f(x)$ is called Lambert's W function.

(a) Write a computer routine that, given x, computes $f(x) = y$ using above definition of y.

(b) Plot $f(x)$ in the interval $0 \leq x \leq 10$ using points spaced at most 0.1 apart.

(c) Compute an approximation to $f'(2)$. Use different values for the step length h until that you think the result is accurate with an error of at most 0.001.

Hint: you are allowed to use program parts of previous homework.

(7 points)