Problem 1 (Eigenvalue distributions). For a string of length $\pi$, the eigenvalues of the Laplacian are $\lambda_n = n^2$ for $n = 1, 2, \ldots$. For a square of length and height $\pi$, they are $\lambda_{k,l} = k^2 + l^2$ for $k = 1, 2, \ldots$ and $l = 1, 2, \ldots$. Finally, in three dimensions, for a box of dimensions $\pi$, the eigenvalues are $\lambda_{k,l,m} = k^2 + l^2 + m^2$.

For each of these three cases, do the following:

- Show the values of the 10 smallest eigenvalues.
- Write a program that calculates all eigenvalues that are smaller than 10,000. Let the program count how many of those are in each of the intervals 0…99, 100…199, 200…299, etc. until 9900…9999. Generate a plot of the number of eigenvalues in each of these bins.

(4 points)

Problem 2 (Eigenfunction expansion). Solve problem 8.3.2 in the book, i.e. derive a solution of the form

$$u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L},$$

with time-dependent coefficients $a_n(t)$. State the differential equations the $a_n(t)$ have to satisfy and derive that the solution must converge to a steady state under the conditions stated in the problem.

(4 points)

Problem 3 (Eigenfunction expansion). Use the method of eigenfunction expansions to solve the following problem:

$$\frac{\partial u(x, t)}{\partial t} - k \frac{\partial^2 u(x, t)}{\partial x^2} = 1,$$

$u(0, t) = 0,$
$u(1, t) = 0,$
$u(x, 0) = 0.$

You may use the formulas from Problem 2, but this time need to solve for the explicit form of the coefficients $a_n(t)$.

(4 points)
Problem 4 (Eigenfunction expansion of a different equation). Consider the following variant of the heat equation (note the additional term in the PDE):

\[
\frac{\partial u(x,t)}{\partial t} - k \frac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t) = q(x),
\]

\[
u(0,t) = 0, \quad u(1,t) = 0, \quad u(x,0) = f(x).
\]

As for the heat equation, the solution can be written as

\[
u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin \frac{n\pi x}{L}.
\]

However, the coefficients \(a_n(t)\) now have to satisfy a different ordinary differential equation. Go back to your notes to see how the ODE for \(a_n(t)\) was derived for the heat equation and adjust this process to the present equation. State which ODE \(a_n(t)\) has to satisfy here. (3 points)