MATH 412: Theory of Partial Differential Equations

Lecturer: Prof. Wolfgang Bangerth
Blocker Bldg., Room 507D
(979) 845 6393
bangerth@math.tamu.edu
http://www.math.tamu.edu/~bangerth

Homework assignment 9 – due Thursday 11/9/2007

Problem 1 (Wave equation). In class, we derived the wave equation for a one-dimensional string under the assumption that all forces were either due to stretching or due to external body forces. Assume that there is an additional source of forces, namely friction that the string experiences in the surrounding medium (for example a string vibrating in air). Let us assume that the force on a piece of string of length Δx is

$$F_{\text{friction}} = -\alpha \, \Delta x \, \frac{\partial u(x,t)}{\partial t},$$

i.e. it is proportional (with constant α) to the length of the piece of the string and to the velocity with which it moves through the surrounding medium; the negative sign indicates that the friction force is in the direction opposite to the string's velocity.

Add this friction force to the two other sources of forces – tension and body forces – and continue the derivation as we did in class. State the form of the equation you arrive at in place of the wave equation. **(4 points)**

Problem 2 (Eigenfunctions of the Laplacian). In class, we have explicitly derived that the eigenfunctions of the Laplacian operator on a rectangle of length L and height H are

$$\phi_n(x,y) = \sin\frac{k\pi x}{L} \sin\frac{l\pi y}{H},$$

where n was a *multi-index* $n = (k,l)$ of two numbers. In the one-dimensional situation, we had the very convenient formula

$$\frac{2}{L}\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} \, dx = \delta_{nm}.$$

Noting that the sines here are the eigenfunctions of the Laplacian in 1-d, one could conjecture that the same holds also for 2-d:

$$\frac{2}{L}\frac{2}{H}\int_0^H \int_0^L \phi_n(x,y)\phi_m(x,y) \, dx \, dy = \delta_{nm}.$$

Prove this formula. Note that n, m are both multi-indices, i.e. for example $n = (k, l), m = (p, q)$ and that δ_{nm} is one only if $n = m$, i.e. if $k = p, l = q$. In other words, $\delta_{nm} = \delta_{kp}\delta_{lq}$. **(4 points)**

Problem 3 (Solutions of the 2d wave equation). Using the eigenfunctions of the Laplacian on the unit square $\Omega = [0, 1]^2$, and the orthogonality formula of Problem 1, compute the solution to the wave equation of the wave equation

$$\begin{aligned} \frac{\partial^2 u(x,y,t)}{\partial t^2} - c^2 \Delta u(x,y,t) &= 0, && \text{in } \Omega \times [0,T], \\ u(x,y,t) &= 0 && \text{for } x \in \partial\Omega, \end{aligned}$$

if we impose initial values

$$\begin{aligned} u(x,y,0) &= 0, \\ \frac{\partial u}{\partial t}(x,y,0) &= \begin{cases} 1 & \text{for } x < \frac{1}{2} \text{ and } y < \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(4 points)

Problem 4 (Eigenvalues of the Laplacian in 3d). In class we derived the eigenfunctions and eigenvectors of the Laplacian on the two-dimensional rectangle of length L and height H. Use the same techniques to derive the eigenfunctions and eigenvalues of the Laplacian in the three-dimensional box of length L, height H, and width W. **(4 points)**