MATH 412: Theory of Partial Differential Equations

Lecturer: Prof. Wolfgang Bangerth
Blocker Bldg., Room 507D
(979) 845 6393
bangerth@math.tamu.edu
http://www.math.tamu.edu/~bangerth

Homework assignment 4 – due Thursday 9/27/2007

Problem 1 (Retake; solutions of the heat equation). Solve problem 2.3.3 (all parts) in the book. Note the remark at the top of the next page and that similar problems are solved in the main text. **(4 points)**

Problem 2 (Solutions of the Laplace equation). In class we have seen that one can solve the Laplace equation on a rectangle

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

by chopping the solution up into four parts $u = u_1 + u_2 + u_3 + u_4$, where each of the u_i satisfies boundary conditions on one part of the boundary and is zero on the other three parts. We have constructed u_4 in class that satisfies the boundary conditions at $x = 0$. Construct u_2 that satisfies the boundary conditions at the right edge, $x = L$, and is zero on all other parts of the boundary. Identify the steps that differ from what we did in computing u_4. **(5 points)**

Problem 3 (Uniqueness of solutions of the Laplace equation). In class, it was shown that the heat equation has solutions (existence) by explicitly constructing them, and that there is only one solution (uniqueness).

We achieved the latter by assuming that there are two solutions u_1, u_2 that satisfy the PDE, boundary conditions, and initial conditions. We then introduced the difference $\delta = u_1 - u_2$ and stated the equations that δ has to satisfy: the homogenous PDE as well as homogenous boundary and initial conditions. If you go back to your notes, you will see that it wasn't particularly hard to derive that $\delta = 0$ by multiplying the PDE for δ, integrating over time and space, and integrating by parts where necessary to arrive at an equation that had two non-negative terms that added up to zero, and therefore proved that each term had to be zero. Thus we have $\delta = 0$ which implies $u_1 = u_2$, which in non-mathematical language reads that every two solutions of the heat equation are equal, or in other words that there is exactly one unique solution.

For the Laplace equation we have shown existence of solutions in class. Prove uniqueness by following the same recipe: assume that there are two solutions

u_1, u_2 of

$$
\begin{aligned}
-\Delta u &= 0 && \text{in } \Omega, \\
u &= g && \text{on } \partial\Omega,
\end{aligned}
$$

introduce the difference $\delta = u_1 - u_2$, derive the equation and boundary conditions δ has to satisfy, then multiply the PDE for δ by δ itself, integrate, and integrate by parts. Show that this implies that $\delta = 0$, i.e. $u_1 = u_2$. **(5 points)**

Problem 4 (Laplacian in a polar coordinate system.) We can identify each point in the plane by specifying its x and y coordinates. Alternatively, we can state its distance r from the origin as well as the angle θ the vector from the origin to the point forms with the positive x-axis. These two coordinate systems are related by

$$
r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan \frac{y}{x}.
$$

If we are given a function $F(r, \theta) = F(r(x, y), \theta(x, y))$, we can compute derivatives with respect to each set of coordinates using the chain rule:

$$
\frac{\partial F(r(x, y), \theta(x, y))}{\partial x} = \frac{\partial F(r, \theta)}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial F(r, \theta)}{\partial \theta} \frac{\partial \theta}{\partial x},
$$

and similarly for $\frac{\partial}{\partial y} F(r(x, y), \theta(x, y))$.

Show that the following identity holds:

$$
\Delta F(r(x, y), \theta(x, y)) = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial}{\partial r} F(r, \theta) \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} F(r, \theta).
$$

Here,

$$
\Delta F(r(x, y), \theta(x, y)) = \frac{\partial^2 F(r(x, y), \theta(x, y))}{\partial x^2} + \frac{\partial^2 F(r(x, y), \theta(x, y))}{\partial y^2}.
$$

The calculations that lead to this result are a bit tedious and lengthy, but since you know the answer you should be able to find a way by simply applying the chain rule often enough and suitably simplifying terms. **(5 points)**