

MATH 417: Numerical Analysis

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Homework assignment 11 – due 11/30/06 and 12/4/06

Problem 1 (Numerical solution of a ODE). Consider the following scalar ordinary differential equation (ODE):

$$x'(t) = \frac{1}{2}x(t), \quad x(0) = 1.$$

The solution of this equation is $x(t) = e^{\frac{1}{2}t}$. Compute approximations to $x(4)$ using the

- first order Taylor expansion method,
- second order Taylor expansion method,
- implicit Euler method,
- trapezoidal method,

each with step sizes $h = 2, 1, \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{32}$. Compute their respective errors $e = |x_N - x(4)|$ where x_N is the approximation to $x(4)$ at the end of the last time step, and compute the convergence rates. Compare the accuracy of all these methods for the same step size h . **(8 points)**

Problem 2 (Numerical solution of a vector-valued ODE). A rocket that is shot up vertically experience upward acceleration from its engines, and downward acceleration due to gravity. Its height therefore satisfies Newton's law

$$d''(t) = \frac{F(t)}{m(t)}, \quad (1)$$

where $d(t)$ denotes the distance from the earth's center. Assume that the rocket is initially at rest at $d(0) = 6371000$. After ignition, the engines produce a constant thrust for 10 minutes before shutting down:

$$T(t) = \begin{cases} 12 & \text{for } t < 600, \\ 0 & \text{for } t \geq 600. \end{cases}$$

On the other hand, gravity generates the force

$$G(t) = -(6371000)^2 \frac{10m(t)}{d(t)^2}.$$

The total force is $F(t) = T(t) + G(t)$. The mass of the rocket decreases while fuel is burnt in the engines according to

$$m(t) = \begin{cases} 1 - \frac{0.9t}{600} & \text{for } t < 600, \\ 0.1 & \text{for } t \geq 600. \end{cases}$$

Rewrite this second order ordinary differential equation as a system of two first order equations. Then numerically approximate the altitude of the rocket for times between $t = 0$ and $t = 36000$ using the explicit Euler method. Try to determine the altitude at $t = 36000$ up to an accuracy of 1000 meters.

(5 points)

Problem 3 (A modeling challenge). *This problem is meant as a bonus question. If you feel bored sitting around the table with your family during Thanksgiving, do what mathematicians typically do on such occasions: scribble solution attempts to questions like this on napkins or the back of envelopes. The problem has two parts: a theoretical part that you can do on the napkin (avoid the linen napkins handed down for generations!), and a practical part where you have to implement your model on a computer. In order to comply with rules on animal experiments, we discourage experimental verification of the model. It won't work with a broiled turkey anyway.*

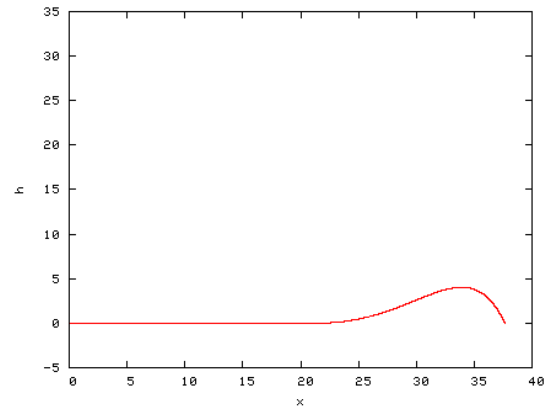
Here's the theoretical problem: Thanksgiving turkeys aren't particularly good at flying. They may try, but they don't really get into the air very gracefully and for extended periods of time. Derive an ODE model for turkey flight that takes into account the following rules (all quantities have units meter, meter per second, etc, as appropriate):

- a) the turkey is initially at rest;
- b) it then runs horizontally, accelerating at a modest rate of 1.5;
- c) when it reaches the lift-off speed of $v = 8$ it gets airborne; from thereon, its vertical (upward) acceleration is $-4 + v$ (in other words, the initial upward acceleration after getting airborne is 4); at the same time, air friction reduces the horizontal velocity by a deceleration of $-v^2/10$;
- d) at some point, the turkey's speed will become too slow to sustain flight, its vertical velocity will become negative, and it will eventually fall back to earth.

To write an ODE model for this, you will have to use the following variables: $x(t)$ —horizontal distance from the starting point; $v(t)$ —horizontal velocity; $h(t)$ —height above ground; $u(t)$ —vertical velocity.

Practical part: Solve these equations from the turkey's start until where it falls back down to earth. Plot $x(t)$ and $h(t)$ in a single plot to show the turkey's trajectory. If you feel challenged, compute the length of the flight in both seconds and meters.

Hint: A plot of $x(t)$ vs. $h(t)$ (i.e. the turkey's trajectory) would look like the figure below. **(6 points)**



Happy Thanksgiving!