Problem 1 (Vector and matrix norms). Solve problems 7.1/1 and 7.1/4. (4 points)

Problem 2 (Equivalence of norms on $\mathbb{R}^n$). In class, we proved the equivalence of the norms $\|\cdot\|_\infty$ and $\|\cdot\|_2$. Here now, prove the same for $\|\cdot\|_\infty$ and $\|\cdot\|_1$, where

\[
\|x\|_1 = \sum_{i=1}^{n} |x_i|.
\]

a) Prove that there are indeed constants $c, C$ such that

\[
c\|v\|_\infty \leq \|v\|_1 \leq C\|v\|_\infty.
\]

where

\[
\|v\|_1 = \sum_i |v_i|, \quad \|v\|_\infty = \max_i |v_i|,
\]

and where $v$ is an $n$-dimensional vector in $\mathbb{R}^n$.

b) For vectors $v_1, v_2$ with $\|v_1\|_1 \leq \|v_2\|_1$, does the result of part a) imply that $\|v_1\|_\infty \leq \|v_2\|_\infty$? If not, give an example of vectors for which this does not follow. (4 points)

Problem 3 (Jacobi iteration). Let $A, b$ be the $100 \times 100$ matrix and 100-dimensional vector defined by

\[
A_{ij} = \begin{cases} 
2.01 & \text{if } i = j, \\
-1 & \text{if } i = j \pm 1, \\
0 & \text{otherwise,}
\end{cases} \quad b_i = \frac{1}{100} \sin \left( \frac{2\pi i}{50} \right).
\]

Apply Jacobi’s method to solving $Ax = b$. Write a program that implements the Jacobi method and start with a vector $x_0$ with randomly chosen elements.
in the range 0 ≤ (x_0)_i ≤ 1 (i.e. with elements generated from what the \texttt{rand()} function or a similar replacement returns).

(Hint: It is not necessary to actually store the complete matrix just to multiply with it. Rather, use that the i-th component of the vector Ay is
\[(Ay)_i = \sum_{j=1}^{n} A_{ij} y_j = 2.01y_i - y_{i-1} - y_{i+1}\] at least for 2 ≤ i ≤ n - 1, and obvious modifications for j = 1 and j = n.)

Run 200 Jacobi iterations and plot the values of (x^{(k)})_i against i for every few iterations, for example \(k = 0, 2, 5, 10, 20, 50, 100, 200\). What do you observe?

\[\text{(5 points)}\]

\textbf{Problem 4 (Alternative vector norms).} Let \(A\) be a symmetric and positive definite \(n \times n\) matrix. Show that
\[\|x\|_A = \sqrt{x^T A x}\]
is a norm for vectors \(x \in \mathbb{R}^n\). (Hint: Use the eigenvalue and eigenvector decomposition of symmetric positive definite matrices.)

\[\text{(3 points)}\]