Problem 1 (Secant method). This problem is an example of finding the root of a function \( f \) that is only given in form of a procedure, a likely case in applications, instead of as a closed form expression.

In order to define the function \( g(x) \), consider the following iteration: set \( a_0 = 1 \) and compute the values \( a_i \) by the following iteration:

\[
a_i = a_{i-1} + \frac{x \sin a_{i-1} + x}{10}.
\]

Clearly, we can compute \( a_1 \) from \( a_0 \) for each value of \( x \). Similarly, we can compute \( a_2 \) from \( a_1 \), and so on. Now, let \( g(x) \) be the function whose value equals \( a_{10} \) for any given value of \( x \).

a) Write a program function that given a value \( x \) returns \( g(x) = a_{10} \) by computing the iteration above.

b) Assume we want to solve the equation \( f(x) = 0 \) where \( f(x) = g(x) - 3 \). State why Newton’s method may be ill-suited for this task.

c) Write a program that finds a root of \( f(x) = g(x) - 3 \) up to 6 digits accuracy using the secant method. (6 points)

Problem 2 (Root finding methods). Compare, in words, the bisection method, Newton’s method, and the secant method with respect to the following criteria: reliability of finding a root of a function, speed of convergence, complexity (i.e., a method is better if it needs fewer evaluations of \( f(x) \) per iteration, or if it only needs function values rather than derivatives). (3 points)
Problem 3 (Gaussian elimination). Solve (on paper, showing the individual steps) the following system of linear equations using Gaussian elimination:

\[
\begin{bmatrix}
1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\
\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\
\frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\
\frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix}
=
\begin{bmatrix}
1 \\
2 \\
3 \\
4
\end{bmatrix}.
\]

Verify that your result is correct.

(The matrix is the example is the so-called Hilbert matrix, with entries \(H_{ij} = \frac{1}{i+j-1}\). It has a number of nasty properties that make it a good testcase for matrix algorithms.)

(3 points)

Problem 4 (Gaussian elimination). Using Gaussian elimination, it is simple to solve the following problem

\[
\begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}.
\]

One would eliminate the occurrence of \(x_1\) in the second equation by subtracting the first from the second equation, arriving at a diagonal matrix.

Describe what happens if the system instead looked like this:

\[
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
1 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}.
\]

Does the algorithm still work? If not, propose a remedy.

(2 points)