# MATH 412: Theory of Partial Differential Equations

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## Homework assignment 10 – due Thursday 11/16/2006

**Problem 1 (Solutions of the heat equation with inhomogenous boundary conditions).** In class, we have seen how one can solve the heat equation with inhomogenous boundary conditions by a) writing the solution $u(x,t)$ as $u(x,t) = u_E(x) + \tilde{u}(x,t)$, b) solving for the equilibrium temperature $u_E(x)$, and c) solving for the remainder $\tilde{u}$, which has to satisfy a homogenous PDE with homogenous boundary conditions.

Use this technique to derive the solution of heat equation on $[0,1]$:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial t^2} &= 0, \\ u(0,t) &= 1, \\ u(1,t) &= 2, \\ u(x,0) &= \sin(\pi x). \end{aligned}$$

**(5 points)**

**Problem 2 (Eigenfunction expansion).** Solve problem 8.3.2 in the book, i.e. derive a solution of the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L},$$

with time-dependent coefficients $a_n(t)$. State the differential equations the $a_n(t)$ have to satisfy and derive that the solution must converge to a steady state under the conditions stated in the problem. **(4 points)**

**Problem 3 (Eigenfunction expansion).** Use the method of eigenfunction expansions to solve the following problem:

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial t^2} &= 1, \\ u(0,t) &= 0, \\ u(1,t) &= 0, \\ u(x,0) &= 0. \end{aligned}$$

You may use the formulas from Problem 2, but this time need to solve for the explicit form of the coefficients $a_n(t)$. **(4 points)**

**Problem 4 (Eigenfunction expansion of a different equation).** Consider the following variant of the heat equation (note the additional term in the PDE):

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} - k\frac{\partial^2 u(x,t)}{\partial x^2} + \alpha u(x,t) &= q(x),\\ u(0,t) &= 0,\\ u(1,t) &= 0,\\ u(x,0) &= f(x).\end{aligned}$$

As for the heat equation, the solution can be written as

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi x}{L}.$$

However, the coefficients $a_n(t)$ now have to satisfy a different ordinary differential equation. Go back to your notes to see how the ODE for $a_n(t)$ was derived for the heat equation and adjust this process to the present equation. State which ODE $a_n(t)$ has to satisfy here. **(3 points)**