Problem 1 (Best polynomial approximation). Compute, analytically (i.e. with exact values, not numerical floating point values), the best polynomial approximation of degree 4 on the interval $[-1, 1]$ to the following functions:

- a) $f(x) = \frac{1}{x}$;
- a) $f(x) = e^x$.

Plot your best approximation $p_4(x)$ together with $f(x)$. \((5 \text{ points})\)

Problem 2 (Gram-Schmidt orthogonalization). Define the following scalar product between matrices $A, B \in \mathbb{R}^{2 \times 2}$:

$$\langle A, B \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} A_{ij} B_{ij},$$

and corresponding norm

$$\|A\|^2 = \langle A, A \rangle = \sum_{i=1}^{2} \sum_{j=1}^{2} A_{ij}^2.$$

Starting with matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

compute four matrices $B_i, 1 \leq i \leq 4$ that are orthonormal onto each other, i.e. for which $\langle B_i, B_j \rangle = \delta_{ij}$ holds. Write the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as

$I = \sum_{i=1}^{4} \beta_i B_i$ and give the coefficients $\beta_i$. \((4 \text{ points})\)
Problem 3 (Least-squares approximation and other norms). In class, we defined the least-square approximating polynomial $p_n(x)$ as that polynomial that minimized the error

$$e_2 = \sum_{i=1}^{N} |p_n(x_i) - y_i|^2,$$

where we used the $l_2$ norm of the difference between $p_n(x)$ and $y$ (i.e. we squared the difference, and summed over it). It was shown that this then leads to a linear problem for finding the expansion coefficients. On the other hand, if we had chosen any other exponent, the problem would have been nonlinear.

Take the same points from last week again:

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_i$</td>
<td>1.51</td>
<td>2.01</td>
<td>2.49</td>
<td>2.98</td>
<td>3.51</td>
<td>4.01</td>
<td>4.49</td>
<td>5.02</td>
<td>5.52</td>
<td>5.98</td>
</tr>
</tbody>
</table>

Find the polynomials $p_1^q(x) = c_0 + c_1x$ that minimize the $l_q$-norms

$$e_q = \sum_{i=1}^{N} |p_n(x_i) - y_i|^q,$$

for $q = 1, q = 2, q = 4$. (For $q = 2$, this is the solution of Problem 4 of last week’s homework.) In addition, compute $p_\infty^1(x) = c_0 + c_1x$ that minimizes the infinity norm

$$e_q = \max_{i \leq i \leq N} |p_n(x_i) - y_i|.$$

Plot the $p_1^q(x)$ together in one plot in which you also show the 10 data points.

Repeat these computations for the following data set (the third to last data point has been changed: some large measurement error has occured, or someone made a mistake transferring the device reading to the data sheet; or maybe this was what the experiment really gave):

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>1</th>
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<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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</tr>
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<tbody>
<tr>
<td>$y_i$</td>
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<td>2.49</td>
<td>2.98</td>
<td>3.51</td>
<td>4.01</td>
<td>4.49</td>
<td>5.82</td>
<td>5.52</td>
<td>5.98</td>
</tr>
</tbody>
</table>

Comment on the suitability of the solutions you’ve found for approximating the two data sets.

Note: To compute each of these polynomials, you have to find the coefficients $c_0, c_1$ that minimize the respective error $e_q$ that can be expressed as a function of $c_i$ by substituting $p_1^q(x) = c_0 + c_1x$. In general, you will not be able to find these coefficients exactly except for the case $q = 2$. In particular, for $q = 1, \infty$ you can’t even find them by looking for points at which $\frac{\partial e_1}{\partial c_0} = \frac{\partial e_1}{\partial c_1} = 0$, since $e_q$ is not differentiable. In this case, feel free to get approximate values of the coefficients by plotting $e_q$ as a function of $c_i$ and visually determining values for which it is minimal.

For $q = 4$, one ends up with an error function $e_q$ that is quartic in $c_i$, i.e. nonlinear but differentiable. Determine its minimum either visually, or by letting your favorite math program find it using a minimum/root finder such as Newton’s method. (7 points)