MATH 609-602: Numerical Methods

Lecturer:	Prof. Wolfgang Bangerth Blocker Bldg., Room 507D (979) 845 6393 bangerth@math.tamu.edu
Teaching Assistant:	Seungil Kim Blocker Bldg., Room 507A (979) 862 3259 sgkim@math.tamu.edu

Homework assignment 8 – due Tuesday 11/1/2005

Problem 1 (Best polynomial approximation). Compute, analytically (i.e. with exact values, not numerical floating point values), the best polynomial approximation of degree 4 on the interval $[-1, 1]$ to the following functions:

a) $f(x) = \frac{1}{x}$;

a) $f(x) = e^x$.

Plot your best approximation $p_4(x)$ together with $f(x)$. **(5 points)**

Problem 2 (Gram-Schmidt orthogonalization). Define the following scalar product between matrices $A, B \in \mathbb{R}^{2\times 2}$:

$$\langle A, B\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} A_{ij}B_{ij},$$

and corresponding norm

$$\|A\|^2 = \langle A, A\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2} A_{ij}^2.$$

Starting with matrices

$$A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad A_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

compute four matrices $B_i, 1 \le i \le 4$ that are orthonormal onto each other, i.e. for which $\langle B_i, B_j\rangle = \delta_{ij}$ holds. Write the identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as $I = \sum_{i=1}^{4} \beta_i B_i$ and give the coefficients β_i. **(4 points)**

Problem 3 (Least-squares approximation and other norms). In class, we defined the least-square approximating polynomial $p_n(x)$ as that polynomial that minimized the error

$$e_2 = \sum_{i=1}^{N} |p_n(x_i) - y_i|^2,$$

where we used the l_2 norm of the difference between $p_n(x_i)$ and y_i (i.e. we squared the difference, and summed over it). It was shown that this then leads to a linear problem for finding the expansion coefficients. On the other hand, if we had chosen any other exponent, the problem would have been nonlinear.

Take the same points from last week again:

x_i	1	2	3	4	5	6	7	8	9	10
y_i	1.51	2.01	2.49	2.98	3.51	4.01	4.49	5.02	5.52	5.98

Find the polynomials $p_1^q(x) = c_0 + c_1 x$ that minimize the l_q-norms

$$e_q = \sum_{i=1}^{N} |p_n(x_i) - y_i|^q,$$

for $q = 1, q = 2, q = 4$. (For $q = 2$, this is the solution of Problem 4 of last week's homework.) In addition, compute $p_1^\infty(x) = c_0 + c_1 x$ that minimizes the infinity norm

$$e_q = \max_{i \le i \le N} |p_n(x_i) - y_i|.$$

Plot the $p_1^q(x)$ together in one plot in which you also show the 10 data points.

Repeat these computations for the following data set (the third to last data point has been changed: some large measurement error has occured, or someone made a mistake transfering the device reading to the data sheet; or maybe this was what the experiment really gave):

x_i	1	2	3	4	5	6	7	8	9	10
y_i	1.51	2.01	2.49	2.98	3.51	4.01	4.49	5.82	5.52	5.98

Comment on the suitability of the solutions you've found for approximating the two data sets.

Note: To compute each of these polynomials, you have to find the coefficients c_0, c_1 that minimize the respective error e_q that can be expressed as a function of c_i by substituting $p_1^q(x) = c_0 + c_1 x$. In general, you will not be able to find these coefficients exactly except for the case $q = 2$. In particular, for $q = 1, \infty$ you can't even find them by looking for points at which $\frac{\partial e_q}{\partial c_0} = \frac{\partial e_q}{\partial c_1} = 0$, since e_q is not differentiable. In this case, feel free to get approximate values of the coefficients by plotting e_q as a function of c_i and visually determining values for which it is minimal.

For $q = 4$, one ends up with an error function e_q that is quartic in c_i, i.e. nonlinear but differentiable. Determine its minimum either visually, or by letting your favorite math program find it using a minimum/root finder such as Newton's method. **(7 points)**