MATH 609-602: Numerical Methods

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Homework assignment 5 – due Tuesday 10/11/2005

Problem 1 (Steepest descent iteration; since the prof's claim that the solution wiggles back and forth didn't seem to be too convincing and he didn't have much of a good explanation anyway :-). The claim was that for badly conditioned matrices the solution vector x_k of iteration k wiggles back and forth, rather than making one step towards the main axis of the contour lines of the quadratic function $q(y)$ and then going straight towards the minimum. Let us test this claim:

Take a matrix and right hand side for a two-dimensional problem as follows:

$$A = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 10 & 0 \end{pmatrix}.$$

The solution of the linear system $Ax = b$ is $x = (1, 0)$. Generate graphs that show the surface and contours of the function

$$q(y) = \frac{1}{2} y^T A y - y^T b.$$

Next consider the steepest descent iteration. Start from $\tilde{x} = (2, 10)$. Perform 100 iterations, where in each iteration you compute

$$g = A\tilde{x} - b, \qquad \alpha = \frac{g^T g}{g^T A g},$$

and then set $\tilde{x} := \tilde{x} - \alpha g$. Plot the iterates $\tilde{x} = (\tilde{x}_1, \tilde{x}_2)$ in a 2-dimensional plot and connect them by lines to see their convergence.

How many iterations do you need to achieve an accuracy of $\|\tilde{x} - x\|_2 \leq 10^4$? Repeat the experiment where a_{11} and b_1 both have the values 1, 10, 100, 1000, 10000 (all other elements of A and b unchanged), and starting from $\tilde{x} = (2, a_{11})$. Create a table with the condition number of these matrices and how many iterations it takes to achieve above accuracy. **(5 points)**

Problem 2 (CG iteration). Take the ever-same 100×100 matrix and 100-dimensional vector defined by

$$A_{ij} = \begin{cases} 2.01 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases} \qquad\qquad b_i = \frac{1}{100} \sin\left(\frac{2\pi i}{50}\right).$$

Implement the Conjugate Gradient algorithm as on page 238 of the book, just above Theorem 3.

Start with a vector x_0 with randomly chosen elements in the range $0 \leq (x_0)_i \leq 1$ (i.e. with elements generated from what the `rand()` function or a similar replacement returns). Run 100 iterations and plot $\|x_N - x_{100}\|$ for these vectors, and graph $(x_N)_i$ against i as in Problem 3 of Homework 4 and as in the test.

If you run the algorithm for 200 iterations, does the solution still change significantly? If not, why? **(6 points)**