MATH 609-602: Numerical Methods

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Homework assignment 3 – due Tuesday 9/20/2005

Problem 1 (Newton's method). For functions $f(x)$ of one variable x, Newton's method almost always converges very quickly (in a matter of a few iterations). This is not always the case for multidimensional problems – in fact, such fast convergence is very rare –, but we can find examples where even in 1d Newton's method converges rather slowly.

Consider finding the zero $x = 1$ of the function

$$f(x) = x^{30} - 1.$$

a) Use Newton's method and start at $x_0 = 20$.

b) Use the secant method and start at $x_0 = 20, x_1 = 19$.

How many iterations do you need to achieve an accuracy of 10^{-8} with both methods? You will observe very slow convergence. Can you explain from the formulas that express the error e_n as a function of e_{n-1} why convergence is so slow? Does the convergence you observe numerically match the theoretical predictions? Plot the error as a function of the iteration number (i.e. plot e_n against n).

Can you give a geometric interpretation of why convergence is so slow? (Hint: think about the curvature of $f(x)$ and what it might have to do with convergence.) **(5 points)**

Problem 2 (Newton's method). For certain functions, Newton's method will always converge in a single step, no matter where we start. What functions are these, and why is a single step enough? (Hint: think about the graphical interpretation of Newton's method, and when it will produce a new iteration that falls exactly onto the true root of the function. Think a second time about the effects of curvature of $f(x)$.) **(2 points)**

Problem 3 (Gaussian elimination). Solve (on paper, showing the individual steps) the following system of linear equations using Gaussian elimination:

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}.$$

Verify that your result is correct.

(The matrix is the example is the so-called Hilbert matrix, with entries $H_{ij} = \frac{1}{i+j-1}$. It has a number of nasty properties that make it a good testcase for matrix algorithms.) **(4 points)**

Problem 4 (Gaussian elimination). Write a computer function that takes a general $n \times n$ matrix A as input and computes its inverse A^{-1} as output. You may, for example, use the representation $A^{-1} = \prod E_i$ of the inverse as the product of elemental matrices, or a more efficient representation.

Apply this function to compute, numerically, the solution of Problem 3.

(4 points)

Problem 5 (Gaussian elimination). Using Gaussian elimination, it is simple to solve the following problem

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

One would eliminate the occurrence of x_1 in the second equation by subtracting the first from the second equation, arriving at a diagonal matrix.

Describe what happens if the system instead looked like this:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Does the algorithm still work? If not, propose a remedy. **(2 points)**

Problem 6 (Multidimensional root finding). Use Newton's method and your function from Problem 4 to find the minimum of

a) $g(x, y) = e^x - \cos(x + y) - \cos(x - 2y)$,

b) $g(x, y) = e^{y-\sin(4x)} + e^{-y+\sin(4x)}$,

in both cases starting at $x_0 = y_0 = 1$. Remember that finding the minimum of $g(x, y)$ amounts to finding a simultaneous root of the system

$$f_1(x, y) = \frac{\partial g(x, y)}{\partial x} = 0,$$
$$f_2(x, y) = \frac{\partial g(x, y)}{\partial y} = 0.$$

What happens if you start at $x_0 = y_0 = 0$ in case b)? **(5 points)**