Math 369 final exam practice

Basics

concepts: *vectors, lines, planes, length, dot product*

1. Find an equation of the line through the points $(0, 1, 2)$ and $(3, 4, 5)$.

2. Find an equation for the plane through the origin perpendicular to the line in Problem 1.

Linear systems

concepts: *Gaussian elimination, (R)REF, linear independence, span*

3. Write an augmented matrix representing the equations

$$\begin{aligned} 2x + y + z &= 6 \\ y + 2z &= 5 \\ 2x - z &= 1. \end{aligned}$$

Find the REF and RREF of this matrix. Use it to find the solution set of this system.

4. Decide whether or not the vectors $\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are linearly independent.

5. Is $\begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$ in the span of the vectors in Problem 4?

Matrices

concepts: *matrix multiplication and addition, matrix inverse, LU factorization*

6. Find the inverse of $A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ by Gaussian elimination. Check your answer by using the formula for inverses of 2×2 matrices.

7. Find the LU factorization of the matrix A in Problem 6. For $b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, solve $Ax = b$ in 2 steps using the LU factorization.

Linear spaces and linear transformations

concepts: *Subspaces, basis, dimension, linear transformations, column space/rank, null space/nullity*

8. Consider the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & -1 \end{pmatrix}$. Find bases of col($A$), row($A$) and null(A).

9. What is the dimension of the solution set to $Ax = b$ when $b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$? When $b = \begin{pmatrix} 6 \\ 5 \\ 1 \end{pmatrix}$?

10. Let $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+1 \\ y \end{pmatrix}$. Explain why T is not linear.

11. Let T be projection in $\mathbb{R}^2$ onto the line $y = x$. Find the matrix A associated with T.

Eigenvalues and eigenvectors

concepts: *eigenvalues/eigenvectors, determinants, characteristic polynomial, multiplicity, similarity, diagonalization, power method*

12. Find the eigenvalues and eigenvectors of A from Problem 11.

13. Find the characteristic polynomial $c_A(\lambda)$ of A from Problem 11.

14. Diagonalize the matrix A from Problem 11.

15. Find the determinant of

$$B = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 0 & 2 \\ 2 & -1 & 3 \end{pmatrix}$$

by Laplace expansion. Check your answer by finding the determinant by row reduction.

16. Give an example of a 2×2 matrix with an eigenvalue whose geometric and algebraic multiplicities are not equal.

Orthogonality

concepts: *orthogonality, orthogonal matrices, orthogonal projections, Gram-Schmidt, QR factorization, SVD, least squares*

17. Let S be the span of $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}$. Find an orthonormal basis of S.

18. Let S be as in Problem 17. Find the orthogonal projections of e_1, e_2 and e_3 onto S.

19. Find the QR factorization of $A = \begin{pmatrix}1 & 0 & 1\\1 & 1 & -1\\0 & 1 & 1\end{pmatrix}$.

20. Find a singular value decomposition of $A = \begin{pmatrix}1 & 0\\1 & 1\\0 & 1\end{pmatrix}$.

21. Let A be as in Problem 20 and $b = \begin{pmatrix}1\\1\\1\end{pmatrix}$. Find the least squares solution[1] to $Ax \approx b$ in three different ways:

(i) by solving $Ax = \text{proj}_S b$, where S is the span of the columns of A
(ii) by using QR factorization
(iii) by using a SVD and pseudoinverse.

[1] that is, the vector x minimizing $\|Ax - b\|^2$