which agrees with the cofactor expansions along the first row.

EXAMPLE 7 A Technique for Evaluating 2 × 2 and 3 × 3 Determinants

$$\begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 4 & -2 \end{vmatrix} = (3)(-2) - (1)(4) = -10$$

$$\begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ -4 & 5 & 6 \\ 7 & -8 & 9 \end{vmatrix} \begin{matrix} 1 & 2 \\ -4 & 5 \\ 7 & -8 \end{matrix}$$

$$= [45 + 84 + 96] - [105 - 48 - 72] = 240$$

Concept Review

- Determinant
- Minor
- Cofactor
- Cofactor expansion

Skills

- Find the minors and cofactors of a square matrix.
- Use cofactor expansion to evaluate the determinant of a square matrix.
- Use the arrow technique to evaluate the determinant of a 2×2 or 3×3 matrix.
- Use the determinant of a 2×2 invertible matrix to find the inverse of that matrix.
- Find the determinant of an upper triangular, lower triangular, or diagonal matrix by inspection.

Exercise Set 2.1

In Exercises 1–2, find all the minors and cofactors of the matrix A.

1. $A = \begin{bmatrix} 1 & -2 & 3 \\ 6 & 7 & -1 \\ -3 & 1 & 4 \end{bmatrix}$

Answer:

$M_{11} = 29,\ C_{11} = 29$
$M_{12} = 21,\ C_{12} = -21$
$M_{13} = 27,\ C_{13} = 27$
$M_{21} = -11,\ C_{21} = 11$
$M_{22} = 13,\ C_{22} = 13$
$M_{23} = -5,\ C_{23} = 5$
$M_{31} = -19,\ C_{31} = -19$
$M_{32} = -19,\ C_{32} = 19$
$M_{33} = 19,\ C_{33} = 19$

2. $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 0 & 1 & 4 \end{bmatrix}$

3. Let

$$A = \begin{bmatrix} 4 & -1 & 1 & 6 \\ 0 & 0 & -3 & 3 \\ 4 & 1 & 0 & 14 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

Find

(a) M_{13} and C_{13}.
(b) M_{23} and C_{23}.
(c) M_{22} and C_{22}.
(d) M_{21} and C_{21}.

Answer:

(a) $M_{13} = 0,\ C_{13} = 0$
(b) $M_{23} = -96,\ C_{23} = 96$
(c) $M_{22} = -48,\ C_{22} = -48$
(d) $M_{21} = 72,\ C_{21} = -72$

4. Let

$$A = \begin{bmatrix} 2 & 3 & -1 & 1 \\ -3 & 2 & 0 & 3 \\ 3 & -2 & 1 & 0 \\ 3 & -2 & 1 & 4 \end{bmatrix}$$

Find

(a) M_{32} and C_{32}.
(b) M_{44} and C_{44}.
(c) M_{41} and C_{41}.
(d) M_{24} and C_{24}.

In Exercises 5–8, evaluate the determinant of the given matrix. If the matrix is invertible, use Equation 2 to find its inverse.

5. $\begin{bmatrix} 3 & 5 \\ -2 & 4 \end{bmatrix}$

Answer:

$$22;\ \begin{bmatrix} \frac{2}{11} & -\frac{5}{22} \\ \frac{1}{11} & \frac{3}{22} \end{bmatrix}$$

6. $\begin{bmatrix} 4 & 1 \\ 8 & 2 \end{bmatrix}$

7. $\begin{bmatrix} -5 & 7 \\ -7 & -2 \end{bmatrix}$

Answer:

$$59;\ \begin{bmatrix} -\frac{2}{59} & -\frac{7}{59} \\ \frac{7}{59} & -\frac{5}{59} \end{bmatrix}$$

8. $\begin{bmatrix} \sqrt{2} & \sqrt{6} \\ 4 & \sqrt{3} \end{bmatrix}$

In Exercises 9–14, use the arrow technique to evaluate the determinant of the given matrix.

9. $\begin{bmatrix} a-3 & 5 \\ -3 & a-2 \end{bmatrix}$

Answer:

$a^2 - 5a + 21$

10. $\begin{bmatrix} -2 & 7 & 6 \\ 5 & 1 & -2 \\ 3 & 8 & 4 \end{bmatrix}$

11. $\begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & -7 \\ 1 & 6 & 2 \end{bmatrix}$

Answer:

-65

12. $\begin{bmatrix} -1 & 1 & 2 \\ 3 & 0 & -5 \\ 1 & 7 & 2 \end{bmatrix}$

13. $\begin{bmatrix} 3 & 0 & 0 \\ 2 & -1 & 5 \\ 1 & 9 & -4 \end{bmatrix}$

Answer:

-123

14. $\begin{bmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{bmatrix}$

In Exercises 15–18, find all values of λ for which $\det(A) = 0$.

15. $A = \begin{bmatrix} \lambda - 2 & 1 \\ -5 & \lambda + 4 \end{bmatrix}$

Answer:

$\lambda = 1 \text{ or } -3$

16. $A = \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 0 & \lambda & 2 \\ 0 & 3 & \lambda - 1 \end{bmatrix}$

17. $A = \begin{bmatrix} \lambda - 1 & 0 \\ 2 & \lambda + 1 \end{bmatrix}$

Answer:

$\lambda = 1 \text{ or } -1$

18. $A = \begin{bmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{bmatrix}$

19. Evaluate the determinant of the matrix in Exercise 13 by a cofactor expansion along

(a) the first row.

(b) the first column.

(c) the second row.

(d) the second column.

(e) the third row.

(f) the third column.

Answer:

$(\text{all parts}) - 123$

20. Evaluate the determinant of the matrix in Exercise 12 by a cofactor expansion along

(a) the first row.

(b) the first column.

(c) the second row.

(d) the second column.

(e) the third row.

(f) the third column.

In Exercises 21–26, evaluate $\det(A)$ by a cofactor expansion along a row or column of your choice.

21. $A = \begin{bmatrix} -3 & 0 & 7 \\ 2 & 5 & 1 \\ -1 & 0 & 5 \end{bmatrix}$

Answer:

-40

22. $A = \begin{bmatrix} 3 & 3 & 1 \\ 1 & 0 & -4 \\ 1 & -3 & 5 \end{bmatrix}$

23. $A = \begin{bmatrix} 1 & k & k^2 \\ 1 & k & k^2 \\ 1 & k & k^2 \end{bmatrix}$

Answer:

0

24. $A = \begin{bmatrix} k+1 & k-1 & 7 \\ 2 & k-3 & 4 \\ 5 & k+1 & k \end{bmatrix}$

25. $A = \begin{bmatrix} 3 & 3 & 0 & 5 \\ 2 & 2 & 0 & -2 \\ 4 & 1 & -3 & 0 \\ 2 & 10 & 3 & 2 \end{bmatrix}$

Answer:

-240

26. $A = \begin{bmatrix} 4 & 0 & 0 & 1 & 0 \\ 3 & 3 & 3 & -1 & 0 \\ 1 & 2 & 4 & 2 & 3 \\ 9 & 4 & 6 & 2 & 3 \\ 2 & 2 & 4 & 2 & 3 \end{bmatrix}$

In Exercises 27–32, evaluate the determinant of the given matrix by inspection.

27. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer:

-1

28. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

29. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 3 & 0 \\ 1 & 2 & 3 & 8 \end{bmatrix}$

Answer:

0

30. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

31. $\begin{bmatrix} 1 & 2 & 7 & -3 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Answer:

6

32. $\begin{bmatrix} -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 40 & 10 & -1 & 0 \\ 100 & 200 & -23 & 3 \end{bmatrix}$

33. Show that the value of the following determinant is independent of θ.

$$\begin{vmatrix} \sin(\theta) & \cos(\theta) & 0 \\ -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) - \cos(\theta) & \sin(\theta) + \cos(\theta) & 1 \end{vmatrix}$$

Answer:

The determinant is $\sin^2\theta + \cos^2\theta = 1$.

34. Show that the matrices

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}$$

commute if and only if

$$\begin{vmatrix} b & a-c \\ e & d-f \end{vmatrix} = 0$$

35. By inspection, what is the relationship between the following determinants?

$$d_1 = \begin{vmatrix} a & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix} \quad \text{and} \quad d_2 = \begin{vmatrix} a+\lambda & b & c \\ d & 1 & f \\ g & 0 & 1 \end{vmatrix}$$

Answer:

$d_2 = d_1 + \lambda$

36. Show that

$$\det(A) = \frac{1}{2}\begin{vmatrix} \operatorname{tr}(A) & 1 \\ \operatorname{tr}(A^2) & \operatorname{tr}(A) \end{vmatrix}$$

for every 2×2 matrix A.

37. What can you say about an *n*th-order determinant all of whose entries are 1? Explain your reasoning.

38. What is the maximum number of zeros that a 3×3 matrix can have without having a zero determinant? Explain your reasoning.

39. What is the maximum number of zeros that a 4×4 matrix can have without having a zero determinant? Explain your reasoning.

40. Prove that (x_1, y_1), (x_2, y_2), and (x_3, y_3) are collinear points if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

41. Prove that the equation of the line through the distinct points (a_1, b_1) and (a_2, b_2) can be written as

$$\begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0$$

42. Prove that if A is upper triangular and B_{ij} is the matrix that results when the ith row and jth column of A are deleted, then B_{ij} is upper triangular if $i < j$.

True-False Exercises

In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) The determinant of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad + bc$.

Answer:

False

(b) Two square matrices A and B can have the same determinant only if they are the same size.

Answer:

False

(c) The minor M_{ij} is the same as the cofactor C_{ij} if and only if $i + j$ is even.

Answer:

True

(d) If A is a 3×3 symmetric matrix, then $C_{ij} = C_{ji}$ for all i and j.

Answer:

True

(e) The value of a cofactor expansion of a matrix A is independent of the row or column chosen for the expansion.

Answer:

True

(f) The determinant of a lower triangular matrix is the sum of the entries along its main diagonal.

Answer:

False

(g) For every square matrix A and every scalar c, we have $\det(cA) = c \det(A)$.

Answer:

False

(h) For all square matrices A and B, we have $\det(A + B) = \det(A) + \det(B)$.

Answer:

False

(i) For every 2×2 matrix A, we have $\det(A^2) = (\det(A))^2$.

Answer:

True