$$Av_1 = \sqrt{\lambda_1}\mathbf{u}_1 = \sigma_1\mathbf{u}_1, \quad A\mathbf{v}_2 = \sqrt{\lambda_2}\mathbf{u}_2 = \sigma_2\mathbf{u}_2, \ldots, \quad A\mathbf{v}_k = \sqrt{\lambda_k}\mathbf{u}_k = \sigma_k\mathbf{u}_k \tag{4}$$

It follows from Theorem 6.3.6 that we can extend this to an orthonormal basis

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}, \ldots, \mathbf{u}_n\}$$

for R^n. Now let U be the orthogonal matrix

$$U = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \cdots \;\; \mathbf{u}_k \;\; \mathbf{u}_{k+1} \;\; \cdots \;\; \mathbf{u}_n]$$

and let Σ be the diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & & 0 \\ & \sigma_2 & & & & & \\ & & \ddots & & & & \\ & & & \sigma_k & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{bmatrix}$$

It follows from 4, and the fact that $A\mathbf{v}_i = 0$ for $i > k$, that

$$\begin{aligned} U\Sigma &= [\sigma_1\mathbf{u}_1 \;\; \sigma_2\mathbf{u}_2 \;\; \cdots \;\; \sigma_k\mathbf{u}_k \;\; 0 \;\; \cdots \;\; 0] \\ &= [A\mathbf{v}_1 \;\; A\mathbf{v}_2 \;\; \cdots \;\; A\mathbf{v}_k \;\; A\mathbf{v}_{k+1} \;\; \cdots \;\; A\mathbf{v}_n] \\ &= AV \end{aligned}$$

which we can rewrite using the orthogonality of V as $A = U\Sigma V^T$.

Concept Review

- Eigenvalue decomposition
- Hessenberg decomposition
- Schur decomposition
- Magnification of roundoff error
- Properties that A and A^TA have in common
- A^TA is orthogonally diagonalizable
- Eigenvalues of A^TA are nonnegative
- Singular values
- Diagonal entries of a matrix that is not square
- Singular value decomposition

Skills

- Find the singular values of an $m \times n$ matrix.
- Find a singular value decomposition of an $m \times n$ matrix.

Exercise Set 9.5

In Exercises 1–4, find the distinct singular values of A.

1. $A=\begin{bmatrix}1 & 2 & 0\end{bmatrix}$

Answer:

$0, \sqrt{5}$

2. $A=\begin{bmatrix}3 & 0\\ 0 & 4\end{bmatrix}$

3. $A=\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}$

Answer:

$\sqrt{5}$

4. $A=\begin{bmatrix}\sqrt{2} & 0\\ 1 & \sqrt{2}\end{bmatrix}$

In Exercises 5–12, find a singular value decomposition of *A*.

5. $A=\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$

Answer:

$$A=\begin{bmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}\sqrt{2} & 0\\ 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

6. $A=\begin{bmatrix}-3 & 0\\ 0 & -4\end{bmatrix}$

7. $A=\begin{bmatrix}4 & 6\\ 0 & 4\end{bmatrix}$

Answer:

$$A=\begin{bmatrix}\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}}\\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\end{bmatrix}\begin{bmatrix}8 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}}\\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}}\end{bmatrix}$$

8. $A=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}$

9. $A=\begin{bmatrix}-2 & 2\\ -1 & 1\\ 2 & -2\end{bmatrix}$

Answer:

$$A=\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} \\ \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} & -\frac{\sqrt{2}}{6} \end{bmatrix}\begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

10. $A=\begin{bmatrix} -2 & -1 & 2 \\ 2 & 1 & -2 \end{bmatrix}$

11. $A=\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}$

Answer:

$$A=\begin{bmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix}\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

12. $A=\begin{bmatrix} 6 & 4 \\ 0 & 0 \\ 4 & 0 \end{bmatrix}$

13. Prove: If *A* is an $m \times n$ matrix, then A^TA and AA^T have the same rank.

14. Prove part (*d*) of Theorem 9.5.1 by using part (*a*) of the theorem and the fact that *A* and A^TA have *n* columns.

15. (a) Prove part (*b*) of Theorem 9.5.1 by first showing that row$\left(A^TA\right)$ is a subspace of row(*A*).

(b) Prove part (*c*) of Theorem 9.5.1 by using part (*b*).

16. Let $T: R^n \to R^m$ be a linear transformation whose standard matrix *A* has the singular value decomposition $A = U\Sigma V^T$, and let $B = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B' = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ be the column vectors of *V* and U, respectively. Show that $\Sigma = [T]_{B',B}$.

17. Show that the singular values of A^TA are the squares of the singular values of A.

18. Show that if $A = U\Sigma V^T$ is a singular value decomposition of A, then *U* orthogonally diagonalizes AA^T.

True-False Exercises

In parts (a)–(g) determine whether the statement is true or false, and justify your answer.

(a) If *A* is an $m \times n$ matrix, then A^TA is an $m \times m$ matrix

Answer:

False

(b) If A is an $m \times n$ matrix, then $A^T A$ is a symmetric matrix.

Answer:

True

(c) If A is an $m \times n$ matrix, then the eigenvalues of $A^T A$ are positive real numbers.

Answer:

False

(d) If A is an $n \times n$ matrix, then A is orthogonally diagonalizable.

Answer:

False

(e) If A is an $m \times n$ matrix, then $A^T A$ is orthogonally diagonalizable.

Answer:

True

(f) The eigenvalues of $A^T A$ are the singular values of A.

Answer:

False

(g) Every $m \times n$ matrix has a singular value decomposition.

Answer:

True