$$\|\mathbf{u}\|_V = \|[\mathbf{u}]_{B'}\| = \|[\mathbf{u}]_B\|$$

or

$$\|\mathbf{u}\|_V = \|[\mathbf{u}]_{B'}\| = \|P[\mathbf{u}]_{B'}\| \tag{6}$$

Now let **x** be any vector in R^n, and let **u** be the vector in V whose coordinate vector with respect to the basis B' is **x**; that is, $[\mathbf{u}]_{B'} = \mathbf{x}$. Thus, from 6,

$$\|\mathbf{u}\| = \|\mathbf{x}\| = \|P\mathbf{x}\|$$

which proves that P is orthogonal.

Concept Review

- Orthogonal matrix
- Orthogonal operator
- Properties of orthogonal matrices.
- Geometric properties of an orthogonal operator
- Properties of transition matrices from one orthonormal basis to another.

Skills

- Be able to identify an orthogonal matrix.
- Know the possible values for the determinant of an orthogonal matrix.
- Find the new coordinates of a point resulting from a rotation of axes.

Exercise Set 7.1

1. (a) Show that the matrix

$$A = \begin{bmatrix} \frac{4}{5} & 0 & -\frac{3}{5} \\ -\frac{9}{25} & \frac{4}{5} & -\frac{12}{25} \\ \frac{12}{25} & \frac{3}{5} & \frac{16}{25} \end{bmatrix}$$

is orthogonal in three ways: by calculating A^TA, by using part (*b*) of Theorem 7.1.1, and by using part (*c*) of Theorem 7.1.1.

(b) Find the inverse of the matrix A in part (a).

Answer:

(b) $\begin{bmatrix} \frac{4}{5} & -\frac{9}{25} & \frac{12}{25} \\ 0 & \frac{4}{5} & \frac{3}{5} \\ -\frac{3}{5} & -\frac{12}{25} & \frac{16}{25} \end{bmatrix}$

2. (a) Show that the matrix

$$A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

is orthogonal.

(b) Let $T: R^3 \to R^3$ be multiplication by the matrix A in part (a). Find $T(\mathbf{x})$ for the vector $\mathbf{x} = (-2, 3, 5)$. Using the Euclidean inner product on R^3, verify that $\|T(\mathbf{x})\| = \|\mathbf{x}\|$.

3. Determine which of the following matrices are orthogonal. For those that are orthogonal, find the inverse.

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$

(d) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(e) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 & 1 \\ 0 & \frac{1}{\sqrt{3}} & \frac{1}{2} & 0 \end{bmatrix}$

Answer:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

(d) $\begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$

(e) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{5}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{6} & -\frac{5}{6} \\ \frac{1}{2} & \frac{1}{6} & -\frac{5}{6} & \frac{1}{6} \end{bmatrix}$

4. Prove that if A is orthogonal, then A^T is orthogonal.

5. Verify that the reflection matrices in Tables Table 1 and Table 2 of Section 4.9 are orthogonal.

6. Let a rectangular $x'y'$-coordinate system be obtained by rotating a rectangular xy-coordinate system counterclockwise through the angle $\theta = 3\pi / 4$.

(a) Find the $x'y'$-coordinates of the point whose xy-coordinates are $(-2, 6)$.

(b) Find the xy-coordinates of the point whose $x'y'$-coordinates are $(5, 2)$.

7. Repeat Exercise 6 with $\theta = \pi / 3$.

Answer:

(a) $\left(-1 + 3\sqrt{3}, 3 + \sqrt{3}\right)$

(b) $\left(\frac{5}{2} - \sqrt{3},\ \frac{5}{2}\sqrt{3} + 1\right)$

8. Let a rectangular $x'y'z'$-coordinate system be obtained by rotating a rectangular xyz-coordinate system counterclockwise about the z-axis (looking down the z-axis) through the angle $\theta = \pi / 4$.

(a) Find the $x'y'z'$-coordinates of the point whose xyz-coordinates are $(-1, 2, 5)$.

(b) Find the xyz-coordinates of the point whose $x'y'z'$-coordinates are $(1, 6, -3)$.

9. Repeat Exercise 8 for a rotation of $\theta = \pi / 3$ counterclockwise about the y-axis (looking along the positive y-axis toward the origin).

Answer:

(a) $\left(-\frac{1}{2} - \frac{5}{2}\sqrt{3}, 2,\ \frac{5}{2} - \frac{1}{2}\sqrt{3}\right)$

(b) $\left(\frac{1}{2} - \frac{3}{2}\sqrt{3}, 6,\ -\frac{3}{2} - \frac{1}{2}\sqrt{3}\right)$

10. Repeat Exercise 8 for a rotation of $\theta = 3\pi / 4$ counterclockwise about the x-axis (looking along the positive x-axis toward the origin).

11. (a) A rectangular $x'y'z'$-coordinate system is obtained by rotating an xyz-coordinate system counterclockwise about the y-axis through an angle θ (looking along the positive y-axis toward the origin). Find a matrix A such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where (x, y, z) and (x', y', z') are the coordinates of the same point in the xyz- and $x'y'z'$-systems, respectively.

(b) Repeat part (a) for a rotation about the x-axis.

Answer:

(a) $A = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

12. A rectangular $x''y''z''$-coordinate system is obtained by first rotating a rectangular xyz-coordinate system 60° counterclockwise about the z-axis (looking down the positive z-axis) to obtain an $x'y'z'$-coordinate system, and then rotating the $x'y'z'$-coordinate system 45°

counterclockwise about the y'-axis (looking along the positive y'-axis toward the origin). Find a matrix A such that

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where (x, y, z) and (x'', y'', z'') are the *xyz*- and $x''y''z''$-coordinates of the same point.

13. What conditions must *a* and *b* satisfy for the matrix

$$\begin{bmatrix} a+b & b-a \\ a-b & b+a \end{bmatrix}$$

to be orthogonal?

Answer:

$a^2 + b^2 = \frac{1}{2}$

14. Prove that a 2×2 orthogonal matrix A has only one of two possible forms:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

where $0 \le \theta < 2\pi$. [*Hint:* Start with a general 2×2 matrix $A = (a_{ij})$, and use the fact that the column vectors form an orthonormal set in R^2.]

15. (a) Use the result in Exercise 14 to prove that multiplication by a 2×2 orthogonal matrix is either a reflection or a reflection followed by a rotation about the *x*-axis.

(b) Prove that multiplication by A is a rotation if $\det(A) = 1$ and that a reflection followed by a rotation if $\det(A) = -1$.

16. Use the result in Exercise 15 to determine whether multiplication by A is a reflection or a reflection followed by a rotation about the *x*-axis. Find the angle of rotation in either case.

(a)

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

17. Find *a*, *b*, and *c* for which the matrix

$$\begin{bmatrix} a & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ b & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ c & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

is orthogonal. Are the values of *a*, *b*, and *c* unique? Explain.

Answer:

The only possibilities are $a = 0,\ b = -\frac{2}{\sqrt{6}},\ c = \frac{1}{\sqrt{3}}$ or $a = 0,\ b = \frac{2}{\sqrt{6}},\ c = -\frac{1}{\sqrt{3}}$.

18. The result in Exercise 15 has an analog for 3×3 orthogonal matrices: It can be proved that multiplication by a 3×3 orthogonal matrix A is a rotation about some axis if $\det(A) = 1$ and is a rotation about some axis followed by a reflection about some coordinate plane if $\det(A) = -1$. Determine whether multiplication by A is a rotation or a rotation followed by a reflection.

(a)

$$A = \begin{bmatrix} \frac{3}{7} & \frac{2}{7} & \frac{6}{7} \\ -\frac{6}{7} & \frac{3}{7} & \frac{2}{7} \\ \frac{2}{7} & \frac{6}{7} & -\frac{3}{7} \end{bmatrix}$$

(b)
$$A = \begin{bmatrix} \frac{2}{7} & \frac{3}{7} & \frac{6}{7} \\ \frac{3}{7} & -\frac{6}{7} & \frac{2}{7} \\ \frac{6}{7} & \frac{2}{7} & -\frac{3}{7} \end{bmatrix}$$

19. Use the fact stated in Exercise 18 and part (*b*) of Theorem 7.1.2 to show that a composition of rotations can always be accomplished by a single rotation about some appropriate axis.

20. Prove the equivalence of statements (*a*) and (*c*) in Theorem 7.1.1.

21. A linear operator on R^2 is called ***rigid*** if it does not change the lengths of vectors, and it is called ***angle preserving*** if it does not change the angle between nonzero vectors.

(a) Name two different types of linear operators that are rigid.

(b) Name two different types of linear operators that are angle preserving.

(c) Are there any linear operators on R^2 that are rigid and not angle preserving? Angle preserving and not rigid? Justify your answer.

Answer:

(a) Rotations about the origin, reflections about any line through the origin, and any combination of these

(b) Rotation about the origin, dilations, contractions, reflections about lines through the origin, and combinations of these

(c) No; dilations and contractions

True-False Exercises

In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) The matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is orthogonal.

Answer:

False

(b) The matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ is orthogonal.

Answer:

False

(c) An $m \times n$ matrix A is orthogonal if $A^T A = I$.

Answer:

False

(d) A square matrix whose columns form an orthogonal set is orthogonal.

Answer:

False

(e) Every orthogonal matrix is invertible.

Answer:

True

(f) If A is an orthogonal matrix, then A^2 is orthogonal and $(\det A)^2 = 1$.

Answer:

True

(g) Every eigenvalue of an orthogonal matrix has absolute value 1.

Answer:

True

(h) If A is a square matrix and $\|A\mathbf{u}\| = 1$ for all unit vectors $\mathbf{u}$, then A is orthogonal.

Answer:

True