Exercise Set 5.2

In Exercises 1–4, show that A and B are not similar matrices.

1. $A=\begin{bmatrix}1 & 1\\ 3 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 0\\ 3 & -2\end{bmatrix}$

Answer:

Possible reason: Determinants are different.

2. $A=\begin{bmatrix}4 & -1\\ 2 & 4\end{bmatrix}, B=\begin{bmatrix}4 & 1\\ 2 & 4\end{bmatrix}$

3. $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 1 & 2\\ 0 & 0 & 1\end{bmatrix}, B=\begin{bmatrix}1 & 2 & 0\\ \frac{1}{2} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Answer:

Possible reason: Ranks are different.

4. $A=\begin{bmatrix}1 & 0 & 1\\ 2 & 0 & 2\\ 3 & 0 & 3\end{bmatrix}, B=\begin{bmatrix}1 & 1 & 0\\ 2 & 2 & 0\\ 0 & 1 & 1\end{bmatrix}$

5. Let A be a 6×6 matrix with characteristic equation $\lambda^2(\lambda-1)(\lambda-2)^3=0$. What are the possible dimensions for eigenspaces of A?

Answer:

$\lambda=0{:}\ 1 \text{ or } 2;\ \ \lambda=1{:}\ 1;\ \ \lambda=2{:}\ 1,\ 2,\ \text{ or } 3$

6. Let

$$A=\begin{bmatrix}4 & 0 & 1\\ 2 & 3 & 2\\ 1 & 0 & 4\end{bmatrix}$$

(a) Find the eigenvalues of A.

(b) For each eigenvalue λ, find the rank of the matrix $\lambda I-A$.

(c) Is A diagonalizable? Justify your conclusion.

In Exercises 7–11, use the method of Exercise 6 to determine whether the matrix is diagonalizable.

7. $\begin{bmatrix}2 & 0\\ 1 & 2\end{bmatrix}$

Answer:

Not diagonalizable

8. $\begin{bmatrix} 2 & -3 \\ 1 & -1 \end{bmatrix}$

9. $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Answer:

Not diagonalizable

10. $\begin{bmatrix} -1 & 0 & 1 \\ -1 & 3 & 0 \\ -4 & 13 & -1 \end{bmatrix}$

11. $\begin{bmatrix} 2 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

Answer:

Not diagonalizable

In Exercises 12–15, find a matrix *P* that diagonalizes *A*, and compute $P^{-1}AP$.

12. $A = \begin{bmatrix} -14 & 12 \\ -20 & 17 \end{bmatrix}$

13. $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

Answer:

$P = \begin{bmatrix} \frac{1}{3} & 0 \\ 1 & 1 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

15. $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Answer:

$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$; $P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

In Exercises 16–21, find the geometric and algebraic multiplicity of each eigenvalue of the matrix *A*, and determine whether *A* is diagonalizable. If *A* is diagonalizable, then find a matrix *P* that diagonalizes *A*, and find $P^{-1}AP$.

16. $A=\begin{bmatrix}19 & -9 & -6\\ 25 & -11 & -9\\ 17 & -9 & -4\end{bmatrix}$

17. $A=\begin{bmatrix}-1 & 4 & -2\\ -3 & 4 & 0\\ -3 & 1 & 3\end{bmatrix}$

Answer:

$P=\begin{bmatrix}1 & 2 & 1\\ 1 & 3 & 3\\ 1 & 3 & 4\end{bmatrix}$; $P^{-1}AP=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 3\end{bmatrix}$

18. $A=\begin{bmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 0 & 1 & 5\end{bmatrix}$

19. $A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 3 & 0 & 1\end{bmatrix}$

Answer:

$P=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -3 & 0 & 1\end{bmatrix}$; $P^{-1}AP=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}$

20. $A=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 1 & 3\end{bmatrix}$

21. $A=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 5 & -5\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$

Answer:

$P=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 1 & -1\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$; $P^{-1}AP=\begin{bmatrix}-2 & 0 & 0 & 0\\ 0 & -2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$

22. Use the method of Example 5 to compute A^{10}, where

$$A=\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}$$

23. Use the method of Example 5 to compute A^{11}, where

$$A = \begin{bmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} -1 & 10237 & -2047 \\ 0 & 1 & 0 \\ 0 & 10245 & -2048 \end{bmatrix}$$

24. In each part, compute the stated power of

$$A = \begin{bmatrix} 1 & -2 & 8 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a) A^{1000} (b) A^{-1000} (c) A^{2301} (d) A^{-2301}

25. Find A^n if n is a positive integer and

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$

Answer:

$$A^n = PD^nP^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & 4^n \end{bmatrix} \begin{bmatrix} \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

26. Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Show that

(a) A is diagonalizable if $(a-d)^2 + 4bc > 0$.

(b) A is not diagonalizable if $(a-d)^2 + 4bc < 0$.

[*Hint:* See Exercise 19 of Section 5.1.]

27. In the case where the matrix A in Exercise 26 is diagonalizable, find a matrix P that diagonalizes A. [*Hint:* See Exercise 20 of Section 5.1.]

Answer:

On possibility is $P = \begin{bmatrix} -b & -b \\ a-\lambda_1 & a-\lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are as in Exercise 20 of Section 5.1.

28. Prove that similar matrices have the same rank.

29. Prove that similar matrices have the same nullity.

30. Prove that similar matrices have the same trace.

31. Prove that if A is diagonalizable, then so is A^k for every positive integer k.

32. Prove that if A is a diagonalizable matrix, then the rank of A is the number of nonzero eigenvalues of A.

33. Suppose that the characteristic polynomial of some matrix A is found to be $p(\lambda) = (\lambda - 1)(\lambda - 3)^2(\lambda - 4)^3$. In each part, answer the question and explain your reasoning.

(a) What can you say about the dimensions of the eigenspaces of A?

(b) What can you say about the dimensions of the eigenspaces if you know that A is diagonalizable?

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set of eigenvectors of A all of which correspond to the same eigenvalue of A, what can you say about the eigenvalue?

Answer:

(a) $\lambda = 1$: dimension $= 1$; $\lambda = 3$: dimension ≤ 2; $\lambda = 4$: dimension ≤ 3

(b) Dimensions will be exactly 1, 2, and 3.

(c) $\lambda = 4$

34. This problem will lead you through a proof of the fact that the algebraic multiplicity of an eigenvalue of an $n \times n$ matrix A is greater than or equal to the geometric multiplicity. For this purpose, assume that λ_0 is an eigenvalue with geometric multiplicity k.

(a) Prove that there is a basis $B = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n\}$ for R^n in which the first k vectors of B form a basis for the eigenspace corresponding to λ_0.

(b) Let P be the matrix having the vectors in B as columns. Prove that the product AP can be expressed as

$$AP = P\begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

[*Hint:* Compare the first k column vectors on both sides.]

(c) Use the result in part (b) to prove that A is similar to

$$C = \begin{bmatrix} \lambda_0 I_k & X \\ 0 & Y \end{bmatrix}$$

and hence that A and C have the same characteristic polynomial.

(d) By considering $\det(\lambda I - C)$, prove that the characteristic polynomial of C (and hence A) contains the factor $(\lambda - \lambda_0)$ at least k times, thereby proving that the algebraic multiplicity of λ_0 is greater than or equal to the geometric multiplicity k.

True-False Exercises

In parts (a)–(h) determine whether the statement is true or false, and justify your answer.

(a) Every square matrix is similar to itself.

Answer:

True

(b) If A, B, and C are matrices for which A is similar to B and B is similar to C, then A is similar to C.

Answer:

True

(c) If *A* and *B* are similar invertible matrices, then A^{-1} and B^{-1} are similar.

Answer:

True

(d) If *A* is diagonalizable, then there is a unique matrix *P* such that $P^{-1}AP$ is diagonal.

Answer:

False

(e) If *A* is diagonalizable and invertible, then A^{-1} is diagonalizable.

Answer:

True

(f) If *A* is diagonalizable, then A^T is diagonalizable.

Answer:

True

(g) If there is a basis for R^n consisting of eigenvectors of an $n \times n$ matrix *A*, then *A* is diagonalizable.

Answer:

True

(h) If every eigenvalue of a matrix *A* has algebraic multiplicity 1, then *A* is diagonalizable.

Answer:

True