Concept Review

- Composition of matrix transformations
- Reflection about the origin
- One-to-one transformation
- Inverse of a matrix operator
- Linearity conditions
- Linear transformation
- Equivalent characterizations of invertible matrices

Skills

- Find the standard matrix for a composition of matrix transformations.
- Determine whether a matrix operator is one-to-one; if it is, then find the inverse operator.
- Determine whether a transformation is a linear transformation.

Exercise Set 4.10

In Exercises 1–2, let T_A and T_B be the operators whose standard matrices are given. Find the standard matrices for and $T_A \circ T_B$.

1. $$A = \begin{bmatrix} 1 & -2 & 0 \\ 4 & 1 & -3 \\ 5 & 2 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 & 3 \\ 5 & 0 & 1 \\ 6 & 1 & 7 \end{bmatrix}$$

Answer:

$$T_B \circ T_A = \begin{bmatrix} 5 & -1 & 21 \\ 10 & -8 & 4 \\ 45 & 3 & 25 \end{bmatrix}, \quad T_A \circ T_B = \begin{bmatrix} -8 & -3 & 1 \\ -5 & -15 & -8 \\ 44 & -11 & 45 \end{bmatrix}$$

2. $$A = \begin{bmatrix} 6 & 3 & -1 \\ 2 & 0 & 1 \\ 4 & -3 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 0 & 4 \\ -1 & 5 & 2 \\ 2 & -3 & 8 \end{bmatrix}$$

3. Let $T_1(x_1, x_2) = (x_1 + x_2, x_1 - x_2)$ and $T_2(x_1, x_2) = (3x_1, 2x_1 + 4x_2)$.

(a) Find the standard matrices for T_1 and T_2.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2))$ and $T_2(T_1(x_1, x_2))$.

Answer:

(a) $T_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $T_2 = \begin{bmatrix} 3 & 0 \\ 2 & 4 \end{bmatrix}$

(b) $T_2 \circ T_1 = \begin{bmatrix} 3 & 3 \\ 6 & -2 \end{bmatrix}$, $T_1 \circ T_2 = \begin{bmatrix} 5 & 4 \\ 1 & -4 \end{bmatrix}$

(c) $T_2(T_1(x_1,\ x_2)) = (3x_1 + 3x_2, 6x_1 - 2x_2)$,

$T_1(T_2(x_1,\ x_2)) = (5x_1 + 4x_2,\ x_1 - 4x_2)$

4. Let $T_1(x_1, x_2, x_3) = (4x_1,\ -2x_1 + x_2,\ -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2,\ -x_3, 4x_1 - x_3)$.

(a) Find the standard matrices for T_1 and T_2.

(b) Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

(c) Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

5. Find the standard matrix for the stated composition in R^2.

(a) A rotation of 90°, followed by a reflection about the line $y = x$.

(b) An orthogonal projection on the *y*-axis, followed by a contraction with factor $k = \frac{1}{2}$.

(c) A reflection about the *x*-axis, followed by a dilation with factor $k = 3$.

Answer:

(a) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 0 \\ 0 & -3 \end{bmatrix}$

6. Find the standard matrix for the stated composition in R^2.

(a) A rotation of 60°, followed by an orthogonal projection on the *x*-axis, followed by a reflection about the line $y = x$.

(b) A dilation with factor $k = 2$, followed by a rotation of 45°, followed by a reflection about the *y*-axis.

(c) A rotation of 15°, followed by a rotation of 105°, followed by a rotation of 60°.

7. Find the standard matrix for the stated composition in R^3.

(a) A reflection about the *yz*-plane, followed by an orthogonal projection on the *xz*-plane.

(b) A rotation of 45° about the *y*-axis, followed by a dilation with factor $k = \sqrt{2}$.

(c) An orthogonal projection on the *xy*-plane, followed by a reflection about the *yz*-plane.

Answer:

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

8. Find the standard matrix for the stated composition in R^3.

(a) A rotation of 30° about the *x*-axis, followed by a rotation of 30° about the *z*-axis, followed by a contraction with factor $k=\frac{1}{4}$.

(b) A reflection about the *xy*-plane, followed by a reflection about the *xz*-plane, followed by an orthogonal projection on the *yz*-plane.

(c) A rotation of 270° about the *x*-axis, followed by a rotation of 90° about the *y*-axis, followed by a rotation of 180° about the *z*-axis.

9. Determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) $T_1: R^2 \to R^2$ is the orthogonal projection on the *x*-axis, and $T_2: R^2 \to R^2$ is the orthogonal projection on the *y*-axis.

(b) $T_1: R^2 \to R^2$ is the rotation through an angle θ_1, and $T_2: R^2 \to R^2$ is the rotation through an angle θ_2.

(c) $T_1: R^2 \to R^2$ is the orthogonal projection on the *x*-axis, and $T_2: R^2 \to R^2$ is the rotation through an angle θ.

Answer:

(a) $T_1 \circ T_2 = T_2 \circ T_1$

(b) $T_1 \circ T_2 = T_2 \circ T_1$

(c) $T_1 \circ T_2 \neq T_2 \circ T_1$

10. Determine whether $T_1 \circ T_2 = T_2 \circ T_1$.

(a) $T_1: R^3 \to R^3$ is a dilation by a factor *k*, and $T_2: R^3 \to R^3$ is the rotation about the *z*-axis through an angle θ.

(b) $T_1: R^3 \to R^3$ is the rotation about the *x*-axis through an angle θ_1, and $T_2: R^3 \to R^3$ is the rotation about the *z*-axis through an angle θ_2.

11. By inspection, determine whether the matrix operator is one-to-one.

(a) the orthogonal projection on the *x*-axis in R^2

(b) the reflection about the *y*-axis in R^2

(c) the reflection about the line $y=x$ in R^2

(d) a contraction with factor $k>0$ in R^2

(e) a rotation about the *z*-axis in R^3

(f) a reflection about the *xy*-plane in R^3

(g) a dilation with factor $k>0$ in R^3

Answer:

(a) Not one-to-one

(b) One-to-one

(c) One-to-one

(d) One-to-one

(e) One-to-one

(f) One-to-one

(g) One-to-one

12. Find the standard matrix for the matrix operator defined by the equations, and use Theorem 4.10.4 to determine whether the operator is one-to-one.

(a) $\begin{aligned} w_1 &= 8x_1 + 4x_2 \\ w_2 &= 2x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= 2x_1 - 3x_2 \\ w_2 &= 5x_1 + x_2 \end{aligned}$

(c) $\begin{aligned} w_1 &= -x_1 + 3x_2 + 2x_3 \\ w_2 &= 2x_1 + 4x_3 \\ w_3 &= x_1 + 3x_2 + 6x_3 \end{aligned}$

(d) $\begin{aligned} w_1 &= x_1 + 2x_2 + 3x_3 \\ w_2 &= 2x_1 + 5x_2 + 3x_3 \\ w_3 &= x_1 + 8x_3 \end{aligned}$

13. Determine whether the matrix operator $T: R^2 \to R^2$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2)$.

(a) $\begin{aligned} w_1 &= x_1 + 2x_2 \\ w_2 &= -x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= 4x_1 - 6x_2 \\ w_2 &= -2x_1 + 3x_2 \end{aligned}$

(c) $\begin{aligned} w_1 &= -x_2 \\ w_2 &= -x_1 \end{aligned}$

(d) $\begin{aligned} w_1 &= 3x_1 \\ w_2 &= -5x_1 \end{aligned}$

Answer:

(a) One-to-one; $\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$; $T^{-1}(w_1, w_2) = \left(\frac{1}{3}w_1 - \frac{2}{3}w_2, \frac{1}{3}w_1 + \frac{1}{3}w_2\right)$

(b) Not one-to-one

(c) One-to-one; $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$; $T^{-1}(w_1, w_2) = (-w_2, -w_1)$

(d) Not one-to-one

14. Determine whether the matrix operator $T: R^3 \to R^3$ defined by the equations is one-to-one; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

(a) $\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$

(b) $\begin{aligned} w_1 &= x_1 - 3x_2 + 4x_3 \\ w_2 &= -x_1 + x_2 + x_3 \\ w_3 &= -2x_2 + 5x_3 \end{aligned}$

(c) $\begin{aligned} w_1 &= x_1 + 4x_2 - x_3 \\ w_2 &= 2x_1 + 7x_2 + x_3 \\ w_3 &= x_1 + 3x_2 \end{aligned}$

(d) $\begin{aligned} w_1 &= x_1 + 2x_2 + x_3 \\ w_2 &= -2x_1 + x_2 + 4x_3 \\ w_3 &= 7x_1 + 4x_2 - 5x_3 \end{aligned}$

15. By inspection, find the inverse of the given one-to-one matrix operator.

(a) The reflection about the *x*-axis in R^2.

(b) The rotation through an angle of $\pi/4$ in R^2.

(c) The dilation by a factor of 3 in R^2.

(d) The reflection about the *yz*-plane in R^3.

(e) The contraction by a factor of $\frac{1}{5}$ in R^3.

Answer:

(a) Reflection about the *x*-axis

(b) Rotation through the angle $-\frac{\pi}{4}$

(c) Contraction by a factor of $\frac{1}{3}$

(d) Reflection about the *yz*-plane

(e) Dilation by a factor of 5

In Exercises 16—17, use Theorem 4.10.2 to determine whether $T: R^2 \to R^2$ is a matrix operator.

16. (a) $T(x, y) = (2x, y)$

(b) $T(x, y) = (x^2, y)$

(c) $T(x, y) = (-y, x)$

(d) $T(x, y) = (x, 0)$

17. (a) $T(x, y) = (2x + y, x - y)$

(b) $T(x, y) = (x + 1, y)$

(c) $T(x, y) = (y, y)$

(d) $T\left(x, y\right) = \left(\sqrt[3]{x}, \sqrt[3]{y}\right)$

Answer:

(a) Matrix operator

(b) Not a matrix operator

(c) Matrix operator

(d) Not a matrix operator

In Exercises 18–19, use Theorem 4.10.2 to determine whether $T: R^3 \to R^2$ is a matrix transformation.

18. (a) $T(x, y, z) = (x, x + y + z)$

(b) $T(x, y, z) = (1, 1)$

19. (a) $T(x, y, z) = (0, 0)$

(b) $T(x, y, z) = (3x - 4y, 2x - 5z)$

Answer:

(a) Matrix transformation

(b) Matrix transformation

20. In each part, use Theorem 4.10.3 to find the standard matrix for the matrix operator from the images of the standard basis vectors.

(a) The reflection operators on R^2 in Table 1 of Section 4.9 .

(b) The reflection operators on R^3 in Table 2 of Section 4.9 .

(c) The projection operators on R^2 in Table 3 of Section 4.9 .

(d) The projection operators on R^3 in Table 4 of Section 4.9 .

(e) The rotation operators on R^2 in Table 5 of Section 4.9 .

(f) The dilation and contraction operators on R^3 in Table 8 of Section 4.9 .

21. Find the standard matrix for the given matrix operator.

(a) $T: R^2 \to R^2$ projects a vector orthogonally onto the x-axis and then reflects that vector about the y-axis.

(b) $T: R^2 \to R^2$ reflects a vector about the line $y = x$ and then reflects that vector about the x-axis.

(c) $T: R^2 \to R^2$ dilates a vector by a factor of 3, then reflects that vector about the line $y = x$, and then projects that vector orthogonally onto the y-axis.

Answer:

(a) $\begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix}$

22. Find the standard matrix for the given matrix operator.

(a) $T: R^3 \to R^3$ reflects a vector about the *xz*-plane and then contracts that vector by a factor of $\frac{1}{5}$.

(b) $T: R^3 \to R^3$ projects a vector orthogonally onto the *xz*-plane and then projects that vector orthogonally onto the *xy*-plane.

(c) $T: R^3 \to R^3$ reflects a vector about the *xy*-plane, then reflects that vector about the *xz*-plane, and then reflects that vector about the *yz*-plane.

23. Let $T_A: R^3 \to R^3$ be multiplication by

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 2 \\ 4 & 5 & -3 \end{bmatrix}$$

and let $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ be the standard basis vectors for R^3. Find the following vectors by inspection.

(a) $T_A(\mathbf{e}_1)$, $T_A(\mathbf{e}_2)$, and $T_A(e_3)$

(b) $T_A(e_1 + e_2 + e_3)$

(c) $T_A(7e_3)$

Answer:

(a) $T_A(\mathbf{e}_1) = (-1, 2, 4),\ T_A(\mathbf{e}_2) = (3, 1, 5),\ T_A(\mathbf{e}_3) = (0, 2, -3)$

(b) $T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = (2, 5, 6)$

(c) $T_A(7\mathbf{e}_3) = (0, 14, -21)$

24. Determine whether multiplication by A is a one-to-one matrix transformation.

(a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 3 & -4 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & -4 \end{bmatrix}$

(c) $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

25. (a) Is a composition of one-to-one matrix transformations one-to-one? Justify your conclusion.

(b) Can the composition of a one-to-one matrix transformation and a matrix transformation that is not one-to-one be one-to-one? Account for both possible orders of composition and justify your conclusion.

Answer:

(a) Yes

(b) Yes

26. Show that $T(x, y) = (0, 0)$ defines a matrix operator on R^2 but $T(x, y) = (1, 1)$ does not.

27. (a) Prove: If $T: R^n \to R^m$ is a matrix transformation, then $T(\mathbf{0}) = \mathbf{0}$; that is, *T* maps the zero vector in R^n into the zero vector in R^m.

(b) The converse of this is not true. Find an example of a function that satisfies $T(0) = 0$ but is not a matrix transformation.

Answer:

(b) $T(x_1,\ x_2) = \left(x_1^2 + x_2^2,\ x_1 x_2\right)$

28. Prove: An $n \times n$ matrix *A* is invertible if and only if the linear system $A\mathbf{x} = \mathbf{w}$ has exactly one solution for every vector $\mathbf{w}$ in R^n for which the system is consistent.

29. Let *A* be an $n \times n$ matrix such that $\det(A) = \mathbf{0}$, and let $T: R^n \to R^n$ be multiplication by *A*.

(a) What can you say about the range of the matrix *T?* Give an example that illustrates your conclusion.

(b) What can you say about the number of vectors that *T* maps into $\mathbf{0}$?

Answer:

(a) The range of *T* is a proper subset of R^n.

(b) *T* must map infinitely many vectors to 0.

30. Prove: If the matrix transformation $T_A: R^n \to R^n$ is one-to-one, then *A* is invertible.

True-False Exercises

In parts (a)–(f) determine whether the statement is true or false, and justify your answer.

(a) If $T: R^n \to R^m$ and $T(\mathbf{0}) = \mathbf{0}$, then *T* is a matrix transformation.

Answer:

False

(b) If $T: R^n \to R^m$ and $T(c_1\mathbf{x} + c_2\mathbf{y}) = c_1 T(\mathbf{x}) + c_2 T(\mathbf{y})$ for all scalars c_1 and c_2 and all vectors $\mathbf{x}$ and $\mathbf{y}$ in R^n, then *T* is a matrix transformation.

Answer:

True

(c) If $T: R^n \to R^m$ is a one-to-one matrix transformation, then there are no distinct vectors $\mathbf{x}$ and $\mathbf{y}$ for which $T(\mathbf{x} - \mathbf{y}) = 0$.

Answer:

True

(d) If $T: R^n \to R^m$ is a matrix transformation and $m > n$, then *T* is one-to-one.

Answer:

False

(e) If $T: R^n \to R^m$ is a matrix transformation and $m = n$, then *T* is one-to-one.

Answer:

False

(f) If $T: R^n \to R^m$ is a matrix transformation and $m < n$, then *T* is one-to-one.

Answer:

False