- Value
- Domain
- Codomain
- Transformation
- Relationships among the fundamental spaces
- Operator
- Matrix transformation
- Matrix operator
- Standard matrix
- Properties of matrix transformations
- Zero transformation
- Identity operator
- Reflection operator
- Projection operator
- Rotation operator
- Rotation matrix
- Rotation equations
- Axis of rotation in 3-space
- Angle of rotation in 3-space
- Expansion operator
- Compression operator
- Shear
- Dilation
- Contraction

Skills

- Find the domain and codomain of a transformation, and determine whether the transformation is linear.
- Find the standard matrix for a matrix transformation.
- Describe the effect of a matrix operator on the standard basis in R^n.

Exercise Set 4.9

In Exercises 1–2, find the domain and codomain of the transformation $T_A(\mathbf{x}) = A\mathbf{x}$.

1. (a) A has size 3×2.

(b) A has size 2×3.

(c) A has size 3×3.

(d) A has size 1×6.

Answer:

(a) Domain: R^2; codomain: R^3

(b) Domain: R^3; codomain: R^2

(c) Domain: R^3; codomain: R^3

(d) Domain: R^6; codomain: R^1

2. (a) A has size 4×5.

(b) A has size 5×4.

(c) A has size 4×4.

(d) A has size 3×1.

3. If $T(x_1, x_2) = (x_1 + x_2, \ -x_2, 3x_1)$, then the domain of T is _________ , the codomain of T is _________ , and the image of $\mathbf{x} = (1, \ -2)$ under T is _________ .

Answer:

$R^2, \ R^3, \ (-1, 2, 3)$

4. If $T(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 - 2x_2)$, then the domain of T is _________ , the codomain of T is _________ , and the image of $\mathbf{x} = (0, \ -1, 4)$ under T is _________ .

5. In each part, find the domain and codomain of the transformation defined by the equations, and determine whether the transformation is linear.

(a)
$$\begin{aligned} w_1 &= 3x_1 - 2x_2 + 4x_3 \\ w_2 &= 5x_1 - 8x_2 + x_3 \end{aligned}$$

(b)
$$\begin{aligned} w_1 &= 2x_1x_2 - x_2 \\ w_2 &= x_1 + 3x_1x_2 \\ w_3 &= x_1 + x_2 \end{aligned}$$

(c)
$$\begin{aligned} w_1 &= 5x_1 - x_2 + x_3 \\ w_2 &= -x_1 + x_2 + 7x_3 \\ w_3 &= 2x_1 - 4x_2 - x_3 \end{aligned}$$

(d)
$$\begin{aligned} w_1 &= x_1^2 - 3x_2 + x_3 - 2x_4 \\ w_2 &= 3x_1 - 4x_2 - x_3^2 + x_4 \end{aligned}$$

Answer:

(a) Linear; $R^3 \to R^2$

(b) Nonlinear; $R^2 \to R^3$

(c) Linear; $R^3 \to R^3$

(d) Nonlinear; $R^4 \to R^2$

6. In each part, determine whether T is a matrix transformation.

(a) $T(x, y) = (2x, y)$

(b) $T(x, y) = (-y, x)$

(c) $T(x, y) = (2x + y, x - y)$

(d) $T(x, y) = (x^2, y)$

(e) $T(x, y) = (x, y + 1)$

7. In each part, determine whether T is a matrix transformation.

(a) $T(x, y, z) = (0, 0)$

(b) $T(x, y, z) = (1, 1)$

(c) $T(x, y, z) = (3x - 4y, 2x - 5z)$

(d) $T\left(x, y, z\right) = \left(y^2, z\right)$

(e) $T(x, y, z) = (y - 1, x)$

Answer:

(a) and (c) are matrix transformations; (b), (d), and (e) are not matrix transformations.

8. Find the standard matrix for the transformation defined by the equations.

(a) $$\begin{aligned} w_1 &= 2x_1 - 3x_2 + x_4 \\ w_2 &= 3x_1 + 5x_2 - x_4 \end{aligned}$$

(b) $$\begin{aligned} w_1 &= 7x_1 + 2x_2 - 8x_3 \\ w_2 &= -x_2 + 5x_3 \\ w_3 &= 4x_1 + 7x_2 - x_3 \end{aligned}$$

(c) $$\begin{aligned} w_1 &= -x_1 + x_2 \\ w_2 &= 3x_1 - 2x_2 \\ w_3 &= 5x_1 - 7x_2 \end{aligned}$$

(d) $$\begin{aligned} w_1 &= x_1 \\ w_2 &= x_1 + x_2 \\ w_3 &= x_1 + x_2 + x_3 \\ w_4 &= x_1 + x_2 + x_3 + x_4 \end{aligned}$$

9. Find the standard matrix for the operator $T: R^3 \to R^3$ defined by

$$\begin{aligned} w_1 &= 3x_1 + 5x_2 - x_3 \\ w_2 &= 4x_1 - x_2 + x_3 \\ w_3 &= 3x_1 + 2x_2 - x_3 \end{aligned}$$

and then calculate $T(-1, 2, 4)$ by directly substituting in the equations and also by matrix multiplication.

Answer:

$$\begin{bmatrix} 3 & 5 & -1 \\ 4 & -1 & 1 \\ 3 & 2 & -1 \end{bmatrix}; T(-1, 2, 4) = (3, -2, -3)$$

10. Find the standard matrix for the operator T defined by the formula.

(a) $T(x_1, x_2) = (2x_1 - x_2, x_1 + x_2)$

(b) $T(x_1, x_2) = (x_1, x_2)$

(c) $T(x_1, x_2, x_3) = (x_1 + 2x_2 + x_3, x_1 + 5x_2, x_3)$

(d) $T(x_1, x_2, x_3) = (4x_1, 7x_2, -8x_3)$

11. Find the standard matrix for the transformation T defined by the formula.

(a) $T(x_1, x_2) = (x_2, -x_1, x_1 + 3x_2, x_1 - x_2)$

(b) $T(x_1, x_2, x_3, x_4) = (7x_1 + 2x_2 - x_3 + x_4, x_2 + x_3, -x_1)$

(c) $T(x_1, x_2, x_3) = (0, 0, 0, 0, 0)$

(d) $T(x_1, x_2, x_3, x_4) = (x_4, x_1, x_3, x_2, x_1 - x_3)$

Answer:

(a) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \\ 1 & 3 \\ 1 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 7 & 2 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$

12. In each part, find $T(x)$, and express the answer in matrix form.

(a) $\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 3 & 1 & 5 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

(c) $\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} -2 & 1 & 4 \\ 3 & 5 & 7 \\ 6 & 0 & -1 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

(d) $\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 4 \\ 7 & 8 \end{bmatrix}$; $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

13. In each part, use the standard matrix for *T* to find $T(\mathbf{x})$; then check the result by calculating $T(\mathbf{x})$ directly.

(a) $T(x_1, x_2) = (-x_1 + x_2, x_2)$; $\mathbf{x} = (-1, 4)$

(b) $T(x_1, x_2, x_3) = (2x_1 - x_2 + x_3, x_2 + x_3, 0)$; $\mathbf{x} = (2, 1, -3)$

Answer:

(a) $T(-1, 4) = (5, 4)$

(b) $T(2, 1, -3) = (0, -2, 0)$

14. Use matrix multiplication to find the reflection of $(-1, 2)$ about

(a) the *x*-axis.

(b) the y-axis.

(c) the line $y = x$.

15. Use matrix multiplication to find the reflection of $(2, -5, 3)$ about

(a) the xy-plane.

(b) the xz-plane.

(c) the yz-plane.

Answer:

(a) $(2, -5, -3)$

(b) (2, 5, 3)

(c) $(-2, -5, 3)$

16. Use matrix multiplication to find the orthogonal projection of $(2, -5)$ on

(a) the x-axis.

(b) the y-axis.

17. Use matrix multiplication to find the orthogonal projection of $(-2, 1, 3)$ on

(a) the xy-plane.

(b) the xz-plane.

(c) the yz-plane.

Answer:

(a) $(-2, 1, 0)$

(b) $(-2, 0, 3)$

(c) (0, 1, 3)

18. Use matrix multiplication to find the image of the vector $(3, -4)$ when it is rotated through an angle of

(a) $\theta = 30^\circ$.

(b) $\theta = -60^\circ$.

(c) $\theta = 45^\circ$.

(d) $\theta = 90^\circ$.

19. Use matrix multiplication to find the image of the vector $(-2, 1, 2)$ if it is rotated

(a) 30° about the x-axis.

(b) 45° about the y-axis.

(c) 90° about the z-axis.

Answer:

(a) $\left(-2, \dfrac{\sqrt{3}-2}{2}, \dfrac{1+2\sqrt{3}}{2}\right)$

(b) $\left(0, 1, 2\sqrt{2}\right)$

(c) $(-1, -2, 2)$

20. Find the standard matrix for the operator that rotates a vector in R^3 through an angle of -60° about

(a) the x-axis.

(b) the y-axis.

(c) the z-axis.

21. Use matrix multiplication to find the image of the vector $(-2, 1, 2)$ if it is rotated

(a) -30° about the x-axis.

(b) -45° about the y-axis.

(c) -90° about the z-axis.

Answer:

(a) $\left(-2, \dfrac{\sqrt{3}+2}{2}, \dfrac{-1+2\sqrt{3}}{2}\right)$

(b) $\left(-2\sqrt{2}, 1, 0\right)$

(c) (1, 2, 2)

22. In R^3 the ***orthogonal projections*** on the x-axis, y-axis, and z-axis are defined by

$$T_1(x, y, z) = (x, 0, 0), \quad T_2(x, y, z) = (0, y, 0),$$
$$T_3(x, y, z) = (0, 0, z)$$

respectively.

(a) Show that the orthogonal projections on the coordinate axes are matrix operators, and find their standard matrices.

(b) Show that if $T: R^3 \to R^3$ is an orthogonal projection on one of the coordinate axes, then for every vector $\mathbf{x}$ in R^3, the vectors $T(\mathbf{x})$ and $\mathbf{x} - T(\mathbf{x})$ are orthogonal.

(c) Make a sketch showing $\mathbf{x}$ and $\mathbf{x} - T(\mathbf{x})$ in the case where T is the orthogonal projection on the x-axis.

23. Use Formula 15 to derive the standard matrices for the rotations about the x-axis, y-axis, and z-axis in R^3.

24. Use Formula 15 to find the standard matrix for a rotation of $\pi / 2$ radians about the axis determined by the vector $\mathbf{v} = (1, 1, 1)$. [Note: Formula 15 requires that the vector defining the axis of rotation have length 1.]

25. Use Formula 15 to find the standard matrix for a rotation of 180° about the axis determined by the vector $\mathbf{v} = (2, 2, 1)$. [Note: Formula 15 requires that the vector defining the axis of rotation have length 1.]

Answer:

$$\begin{bmatrix} -\frac{1}{9} & \frac{8}{9} & \frac{4}{9} \\ \frac{8}{9} & -\frac{1}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} & -\frac{7}{9} \end{bmatrix}$$

26. It can be proved that if A is a 2×2 matrix with orthonormal column vectors and for which $\det(A) = 1$, then multiplication by A is a rotation through some angle θ. Verify that

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

satisfies the stated conditions and find the angle of rotation.

27. The result stated in Exercise 26 can be extended to R^3; that is, it can be proved that if A is a 3×3 matrix with orthonormal column vectors and for which $\det(A) = 1$, then multiplication by A is a rotation about some axis through some angle θ. Use Formula 15 to show that the angle of rotation satisfies the equation

$$\cos\theta = \frac{\operatorname{tr}(A) - 1}{2}$$

28. Let A be a 3×3 matrix (other than the identity matrix) satisfying the conditions stated in Exercise 27. It can be shown that if $\mathbf{x}$ is any nonzero vector in R^3, then the vector $\mathbf{u} = A\mathbf{x} + A^T\mathbf{x} + \left[1 - \operatorname{tr}\left(A\right)\right]\mathbf{x}$ determines an axis of rotation when $\mathbf{u}$ is positioned with its initial point at the origin. [See "The Axis of Rotation: Analysis, Algebra, Geometry," by Dan Kalman, *Mathematics Magazine*, Vol. 62, No. 4, October 1989.]

(a) Show that multiplication by

$$A = \begin{bmatrix} \frac{1}{9} & -\frac{4}{9} & \frac{8}{9} \\ \frac{8}{9} & \frac{4}{9} & \frac{1}{9} \\ -\frac{4}{9} & \frac{7}{9} & \frac{4}{9} \end{bmatrix}$$

is a rotation.

(b) Find a vector of length 1 that defines an axis for the rotation.

(c) Use the result in Exercise 27 to find the angle of rotation about the axis obtained in part (b).

29. In words, describe the geometric effect of multiplying a vector $\mathbf{x}$ by the matrix A.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$

Answer:

(a) Twice the orthogonal projection on the x-axis.

(b) Twice the reflection about the x-axis.

30. In words, describe the geometric effect of multiplying a vector $\mathbf{x}$ by the matrix A.

(a) $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

(b) $A = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$

31. In words, describe the geometric effect of multiplying a vector $\mathbf{x}$ by the matrix

$$A = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}$$

Answer:

Rotation through the angle 2θ.

32. If multiplication by A rotates a vector $\mathbf{x}$ in the xy-plane through an angle θ, what is the effect of multiplying $\mathbf{x}$ by A^T? Explain your reasoning.

33. Let $\mathbf{x}_0$ be a nonzero column vector in R^2, and suppose that $T: R^2 \to R^2$ is the transformation defined by the formula $T(\mathbf{x}) = \mathbf{x}_0 + R_\theta \mathbf{x}$, where R_θ is the standard matrix of the rotation of R^2 about the origin through the angle θ. Give a geometric description of this transformation. Is it a matrix transformation? Explain.

Answer:

Rotation through the angle θ and translation by $\mathbf{x}_0$; not a matrix transformation since $\mathbf{x}_0$ is nonzero.

34. A function of the form $f(x) = mx + b$ is commonly called a "linear function" because the graph of $y = mx + b$ is a line. Is f a matrix transformation on R?

35. Let $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ be a line in R^n, and let $T: R^n \to R^n$ be a matrix operator on R^n. What kind of geometric object is the image of this line under the operator T? Explain your reasoning.

Answer:

A line in R^n.

True-False Exercises

In parts (a)–(i) determine whether the statement is true or false, and justify your answer.

(a) If A is a 2×3 matrix, then the domain of the transformation T_A is R^2.

Answer:

False

(b) If A is an $m \times n$ matrix, then the codomain of the transformation T_A is R^n.

Answer:

False

(c) If $T: R^n \to R^m$ and $T(0) = 0$, then T is a matrix transformation.

Answer:

False

(d) If $T: R^n \to R^m$ and $T(c_1\mathbf{x} + c_2\mathbf{y}) = c_1T(\mathbf{x}) + c_2T(\mathbf{y})$ for all scalars c_1 and c_2 and all vectors $\mathbf{x}$ and $\mathbf{y}$ in R^n, then T is a matrix transformation.

Answer:

True

(e) There is only one matrix transformation $T: R^n \to R^m$ such that $T(-\mathbf{x}) = -T(\mathbf{x})$ for every vector $\mathbf{x}$ in R^n.

Answer:

False

(f) There is only one matrix transformation $T: R^n \to R^m$ such that $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}-\mathbf{y})$ for all vectors $\mathbf{x}$ and $\mathbf{y}$ in R^n.

Answer:

True

(g) If $\mathbf{b}$ is a nonzero vector in R^n, then $T(\mathbf{x}) = \mathbf{x}+\mathbf{b}$ is a matrix operator on R^n.

Answer:

False

(h) The matrix $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ is the standard matrix for a rotation.

Answer:

False

(i) The standard matrices of the reflections about the coordinate axes in 2-space have the form $\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$, where $a = \pm 1$.

Answer:

True