Concept Review

- Basis
- Standard bases for R^n, P_n, M_{mn}
- Finite-dimensional
- Infinite-dimensional
- Coordinates
- Coordinate vector

Skills

- Show that a set of vectors is a basis for a vector space.
- Find the coordinates of a vector relative to a basis.
- Find the coordinate vector of a vector relative to a basis.

Exercise Set 4.4

1. In words, explain why the following sets of vectors are *not* bases for the indicated vector spaces.

(a) $\mathbf{u}_1 = (1, 2), \mathbf{u}_2 = (0, 3), \mathbf{u}_3 = (2, 7)$ for R^2

(b) $\mathbf{u}_1 = (-1, 3, 2), \mathbf{u}_2 = (6, 1, 1)$ for R^3

(c) $\mathbf{p}_1 = 1 + x + x^2, \mathbf{p}_2 = x - 1$ for P_2

(d) $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix}, C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix}, D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}, E = \begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix}$, for M_{22}

Answer:

(a) A basis for R^2 has two linearly independent vectors.

(b) A basis for R^3 has three linearly independent vectors.

(c) A basis for P_2 has three linearly independent vectors.

(d) A basis for M_{22} has four linearly independent vectors.

2. Which of the following sets of vectors are bases for R^2?

(a) $\{(2, 1), (3, 0)\}$

(b) $\{(4, 1), (-7, -8)\}$

(c) $\{(0, 0), (1, 3)\}$

(d) $\{(3, 9), (-4, -12)\}$

3. Which of the following sets of vectors are bases for R^3?

(a) $\{(1, 0, 0), (2, 2, 0), (3, 3, 3)\}$

(b) $\{(3, 1, -4), (2, 5, 6), (1, 4, 8)\}$

(c) $\{(2, -3, 1), (4, 1, 1), (0, -7, 1)\}$

(d) $\{(1, 6, 4), (2, 4, -1), (-1, 2, 5)\}$

Answer:

(a), (b)

4. Which of the following form bases for P_2?

(a) $1-3x+2x^2, \quad 1+x+4x^2, \quad 1-7x$

(b) $4+6x+x^2, \quad -1+4x+2x^2, \quad 5+2x-x^2$

(c) $1+x+x^2, \quad x+x^2, \quad x^2$

(d) $-4+x+3x^2, \quad 6+5x+2x^2, \quad 8+4x+x^2$

5. Show that the following matrices form a basis for M_{22}.

$$\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

6. Let V be the space spanned by $\mathbf{v}_1 = \cos^2 x, \mathbf{v}_2 = \sin^2 x, \mathbf{v}_3 = \cos 2x$.

(a) Show that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for V.

(b) Find a basis for V.

7. Find the coordinate vector of **w** relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ for R^2.

(a) $\mathbf{u}_1 = (1, 0), \mathbf{u}_2 = (0, 1); \mathbf{w} = (3, -7)$

(b) $\mathbf{u}_1 = (2, -4), \mathbf{u}_2 = (3, 8); \mathbf{w} = (1, 1)$

(c) $\mathbf{u}_1 = (1, 1), \mathbf{u}_2 = (0, 2); \mathbf{w} = (a, b)$

Answer:

(a) $(\mathbf{w})_S = (3, -7)$

(b) $(\mathbf{w})_S = \left(\frac{5}{28}, \frac{3}{14}\right)$

(c) $(\mathbf{w})_S = \left(a, \frac{b-a}{2}\right)$

8. Find the coordinate vector of **w** relative to the basis $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ of R^2.

(a) $\mathbf{u}_1 = (1, -1), \mathbf{u}_2 = (1, 1); \mathbf{w} = (1, 0)$

(b) $\mathbf{u}_1 = (1, -1), \mathbf{u}_2 = (1, 1); \mathbf{w} = (0, 1)$

(c) $\mathbf{u}_1 = (1, -1), \mathbf{u}_2 = (1, 1); \mathbf{w} = (1, 1)$

9. Find the coordinate vector of **v** relative to the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) $\mathbf{v} = (2, -1, 3); \mathbf{v}_1 = (1, 0, 0), \mathbf{v}_2 = (2, 2, 0), \mathbf{v}_3 = (3, 3, 3)$

(b) $\mathbf{v} = (5, -12, 3); \mathbf{v}_1 = (1, 2, 3), \mathbf{v}_2 = (-4, 5, 6), \mathbf{v}_3 = (7, -8, 9)$

Answer:

(a) $(\mathbf{v})_S = (3, -2, 1)$

(b) $(\mathbf{v})_S = (-2, 0, 1)$

10. Find the coordinate vector of $\mathbf{p}$ relative to the basis $S = \{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$.

(a) $\mathbf{p} = 4 - 3x + x^2$; $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = x$, $\mathbf{p}_3 = x^2$

(b) $\mathbf{p} = 2 - x + x^2$; $\mathbf{p}_1 = 1 + x$, $\mathbf{p}_2 = 1 + x^2$, $\mathbf{p}_3 = x + x^2$

11. Find the coordinate vector of A relative to the basis $S = \{A_1, A_2, A_3, A_4\}$.

$$A = \begin{bmatrix} 2 & 0 \\ -1 & 3 \end{bmatrix}; \quad A_1 = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Answer:

$(A)_S = (-1, 1, \ -1, 3)$

In Exercises 12–13, show that $\{A_1, A_2, A_3, A_4\}$ is a basis for M_{22}, and express A as a linear combination of the basis vectors.

12. $A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}; \quad A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$

13. $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}; \quad A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Answer:

$A = A_1 - A_2 + A_3 - A_4$

In Exercises 14–15, show that $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is a basis for P_2, and express p as a linear combination of the basis vectors.

14. $\mathbf{p}_1 = 1 + 2x + x^2$, $\mathbf{p}_2 = 2 + 9x$, $\mathbf{p}_3 = 3 + 3x + 4x^2$; $\mathbf{p} = 2 + 17x - 3x^2$

15. $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x + x^2$, $\mathbf{p}_3 = x^2$; $\mathbf{p} = 7 - x + 2x^2$

Answer:

$\mathbf{p} = 7\mathbf{p}_1 - 8\mathbf{p}_2 + 3\mathbf{p}_3$

16. The accompanying figure shows a rectangular *xy*-coordinate system and an $x'y'$-coordinate system with skewed axes. Assuming that 1-unit scales are used on all the axes, find the $x'y'$-coordinates of the points whose *xy*-coordinates are given.

(a) (1, 1)

(b) (1, 0)

(c) (0, 1)

(d) (*a b*)

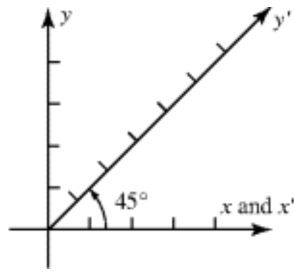

Figure Ex-16

17. The accompanying figure shows a rectangular xy-coordinate system determined by the unit basis vectors **i** and **j** and an $x'y'$-coordinate system determined by unit basis vectors $\mathbf{u}_1$ and $\mathbf{u}_2$. Find the $x'y'$-coordinates of the points whose xy-coordinates are given.

(a) $(\sqrt{3}, 1)$

(b) (1, 0)

(c) (0, 1)

(d) (a, b)

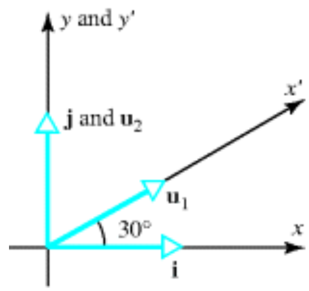

Figure Ex-17

Answer:

(a) (2, 0)

(b) $\left(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

(c) (0, 1)

(d) $\left(\frac{2}{\sqrt{3}}a, b - \frac{a}{\sqrt{3}}\right)$

18. The basis that we gave for M_{22} in Example 4 consisted of noninvertible matrices. Do you think that there is a basis for M_{22} consisting of invertible matrices? Justify your answer.

19. Prove that R^{∞} is infinite-dimensional.

True-False Exercises

In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) If $V = \text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V.

Answer:

False

(b) Every linearly independent subset of a vector space V is a basis for V.

Answer:

False

(c) If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then every vector in V can be expressed as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$

Answer:

True

(d) The coordinate vector of a vector $\mathbf{x}$ in R^n relative to the standard basis for R^n is $\mathbf{x}$.

Answer:

True

(e) Every basis of P_4 contains at least one polynomial of degree 3 or less.

Answer:

False