By Axiom 5 the vector $0\mathbf{u}$ has a negative, $-0\mathbf{u}$. Adding this negative to both sides above yields

$$[0\mathbf{u}+0\mathbf{u}]+(-0\mathbf{u})=0\mathbf{u}+(-0\mathbf{u})$$

or

$$\begin{aligned} 0\mathbf{u}+[0\mathbf{u}+(-0\mathbf{u})]&=0\mathbf{u}+(-0\mathbf{u}) && \textbf{[Axiom 3]}\\ 0\mathbf{u}+\mathbf{0}&=\mathbf{0} && \textbf{[Axiom 5]}\\ 0\mathbf{u}&=\mathbf{0} && \textbf{[Axiom 4]} \end{aligned}$$

Proof (c) To prove that $(-1)\mathbf{u}=-\mathbf{u}$, we must show that $\mathbf{u}+(-1)\mathbf{u}=\mathbf{0}$. The proof is as follows:

$$\begin{aligned} \mathbf{u}+(-1)\mathbf{u}&=1\mathbf{u}+(-1)\mathbf{u} && \textbf{[Axiom 10]}\\ &=(1+(-1))\mathbf{u} && \textbf{[Axiom 8]}\\ &=0\mathbf{u} && \textbf{[Property of numbers]}\\ &=\mathbf{0} && \textbf{[Part (}\boldsymbol{a}\textbf{) of this theorem]} \end{aligned}$$

A Closing Observation

This section of the text is very important to the overall plan of linear algebra in that it establishes a common thread between such diverse mathematical objects as geometric vectors, vectors in R^n, infinite sequences, matrices, and real-valued functions, to name a few. As a result, whenever we discover a new theorem about general vector spaces, we will at the same time be discovering a theorem about geometric vectors, vectors in R^n, sequences, matrices, real-valued functions, and about any new kinds of vectors that we might discover.

To illustrate this idea, consider what the rather innocent-looking result in part (*a*) of Theorem 4.1.1 says about the vector space in Example 8. Keeping in mind that the vectors in that space are positive real numbers, that scalar multiplication means numerical exponentiation, and that the zero vector is the number 1, the equation

$$0\mathbf{u}=\mathbf{0}$$

is a statement of the fact that if u is a positive real number, then

$$u^0=1$$

Concept Review

- Vector space
- Closure under addition
- Closure under scalar multiplication
- Examples of vector spaces

Skills

- Determine whether a given set with two operations is a vector space.
- Show that a set with two operations is not a vector space by demonstrating that at least one of the vector space axioms fails.

Exercise Set 4.1

1. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2), \; k\mathbf{u} = (0, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\boldsymbol{u}$ for $\mathbf{u} = (-1, 2)$, $\mathbf{v} = (3, 4)$ and $k = 3$.

(b) In words, explain why V is closed under addition and scalar multiplication.

(c) Since addition on V is the standard addition operation on R^2, certain vector space axioms hold for V because they are known to hold for R^2. Which axioms are they?

(d) Show that Axioms 7, 8, and 9 hold.

(e) Show that Axiom 10 fails and hence that V is not a vector space under the given operations.

Answer:

(a) $\mathbf{u} + \mathbf{v} = (2, 6)$, $3\mathbf{u} = (0, 6)$

(c) Axioms 1–5

2. Let V be the set of all ordered pairs of real numbers, and consider the following addition and scalar multiplication operations on $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$:

$$\mathbf{u} + \mathbf{v} = (u_1 + v_1 + 1, u_2 + v_2 + 1), \quad k\mathbf{u} = (ku_1, ku_2)$$

(a) Compute $\mathbf{u} + \mathbf{v}$ and $k\boldsymbol{u}$ for $u = (0, 4)$, $v = (1, -3)$, and $k = 2$.

(b) Show that $(0, 0) \neq \mathbf{0}$.

(c) Show that $(-1, -1) = 0$.

(d) Show that Axiom 5 holds by producing an ordered pair $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = 0$ for $\mathbf{u} = (u_1, u_2)$.

(e) Find two vector space axioms that fail to hold.

In Exercises 3–12, determine whether each set equipped with the given operations is a vector space. For those that are not vector spaces identify the vector space axioms that fail.

3. The set of all real numbers with the standard operations of addition and multiplication.

Answer:

The set is a vector space with the given operations.

4. The set of all pairs of real numbers of the form $(x, 0)$ with the standard operations on R^2.

5. The set of all pairs of real numbers of the form (x, y), where $x \geq 0$, with the standard operations on R^2.

Answer:

Not a vector space, Axioms 5 and 6 fail.

6. The set of all n-tuples of real numbers that have the form $(x, x, \ldots, x)$ with the standard operations on R^n.

7. The set of all triples of real numbers with the standard vector addition but with scalar multiplication defined by

$$k(x, y, z) = \left(k^2x, k^2y, k^2z\right)$$

Answer:

Not a vector space. Axiom 8 fails.

8. The set of all 2×2 invertible matrices with the standard matrix addition and scalar multiplication.

9. The set of all 2×2 matrices of the form

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

with the standard matrix addition and scalar multiplication.

Answer:

The set is a vector space with the given operations.

10. The set of all real-valued functions *f* defined everywhere on the real line and such that $f(1) = 0$ with the operations used in Example 6.

11. The set of all pairs of real numbers of the form (1, *x*) with the operations

$$(1, y) + (1, y') = (1, y + y') \text{ and } k(1, y) = (1, ky)$$

Answer:

The set is a vector space with the given operations.

12. The set of polynomials of the form $a_0 + a_1x$ with the operations

$$(a_0 + a_1x) + (b_0 + b_1x) = (a_0 + b_0) + (a_1 + b_1)x$$

and

$$k(a_0 + a_1x) = (ka_0) + (ka_1)x$$

13. Verify Axioms 3, 7, 8, and 9 for the vector space given in Example 4.

14. Verify Axioms 1, 2, 3, 7, 8, 9, and 10 for the vector space given in Example 6.

15. With the addition and scalar multiplication operations defined in Example 7, show that $V = R^2$ satisfies Axioms 1-9.

16. Verify Axioms 1, 2, 3, 6, 8, 9, and 10 for the vector space given in Example 8.

17. Show that the set of all points in R^2 lying on a line is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the line passes through the origin.

18. Show that the set of all points in R^3 lying in a plane is a vector space with respect to the standard operations of vector addition and scalar multiplication if and only if the plane passes through the origin.

In Exercises 19–21, prove that the given set with the stated operations is a vector space.

19. The set $V = \{\mathbf{0}\}$ with the operations of addition and scalar multiplication given in Example 1.

20. The set R^∞ of all infinite sequences of real numbers with the operations of addition and scalar multiplication given in Example 3.

21. The set M_{mn} of all $m \times n$ matrices with the usual operations of addition and scalar multiplication.

22. Prove part (*d*) of Theorem 4.1.1.

23. The argument that follows proves that if **u**, **v**, and **w** are vectors in a vector space *V* such that $\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$, then $\mathbf{u} = \mathbf{v}$ (the ***cancellation law*** for vector addition). As illustrated, justify the steps by filling in the blanks.

$\mathbf{u} + \mathbf{w} = \mathbf{v} + \mathbf{w}$	Hypothesis
$(\mathbf{u} + \mathbf{w}) + (-\mathbf{w}) = (\mathbf{v} + \mathbf{w}) + (-\mathbf{w})$	Add $-\mathbf{w}$ to both sides.
$\mathbf{u} + [\mathbf{w} + (-\mathbf{w})] = \mathbf{v} + [\mathbf{w} + (-\mathbf{w})]$	________________
$\mathbf{u} + \mathbf{0} = \mathbf{v} + \mathbf{0}$	________________
$\mathbf{u} = \mathbf{v}$	________________

24. Let **v** be any vector in a vector space *V*. Prove that $0\mathbf{v} = \mathbf{0}$.

25. Below is a seven-step proof of part (*b*) of Theorem 4.1.1. Justify each step either by stating that it is true by *hypothesis* or by specifying which of the ten vector space axioms applies.

Hypothesis: Let **u** be any vector in a vector space *V*, let 0 be the zero vector in *V*, and let *k* be a scalar.

Conclusion: Then $k\mathbf{0} = \mathbf{0}$.

Proof:

(1) $k\mathbf{0} + k\mathbf{u} = k(\mathbf{0} + \mathbf{u}$

(2) $= k\mathbf{u}$

(3) Since $k\mathbf{u}$ is in V, $-k\mathbf{u}$ is in V.

(4) Therefore, $(k\mathbf{0} + k\mathbf{u} + (-k\mathbf{u} = k\mathbf{u} + (-k\mathbf{u})$.

(5) $k\mathbf{0} + (k\mathbf{u} + (-k\mathbf{u})) = k\mathbf{u} + (-k\mathbf{u})$

(6) $k\mathbf{0} + \mathbf{0} = \mathbf{0}$

(7) $k\mathbf{0} = \mathbf{0}$

26. Let **v** be any vector in a vector space V. Prove that $-\mathbf{v} = (-1)\mathbf{v}$.

27. Prove: If **u** is a vector in a vector space V and k a scalar such that $k\mathbf{u} = \mathbf{0}$, then either $k = 0$ or $\mathbf{u} = 0$. [*Suggestion:* Show that if $k\mathbf{u} = \mathbf{0}$ and $k \neq 0$, then $\mathbf{u} = \mathbf{0}$. The result then follows as a logical consequence of this.]

True-False Exercises

In parts (a)–(e) determine whether the statement is true or false, and justify your answer.

(a) A vector is a directed line segment (an arrow).

Answer:

False

(b) A vector is an n-tuple of real numbers.

Answer:

False

(c) A vector is any element of a vector space.

Answer:

True

(d) There is a vector space consisting of exactly two distinct vectors.

Answer:

False

(e) The set of polynomials with degree exactly 1 is a vector space under the operations defined in Exercise 12.

Answer:

False