1 (9 points)

(a) Compute $\cot(2\pi/3) + e^{\ln(5)}$.

Solution.

$$\begin{aligned}\cot(2\pi/3) + e^{\ln(5)} &= \frac{1}{\tan(2\pi/3)} + 5 \\ &= \frac{1}{-\sqrt{3}} + 5 \\ &= -\frac{1}{\sqrt{3}} + 5.\end{aligned}$$

(b) Find all values of x that satisfy $2 \leq x^2 - 2 < 7$.

Solution.

$$\begin{aligned}2 \leq x^2 - 2 < 7 &\Longrightarrow 4 \leq x^2 < 9 \\ &\Longrightarrow 2 \leq x < 3 \text{ or } -3 < x \leq -2.\end{aligned}$$

(c) What is f(g(h(x))), where $f(x) = x^2 + 1,\ g(x) = 1 - 2x,$ and $h(x) = \sin(x)$?

Solution.

$$\begin{aligned}f(g(h(x))) &= f(g(\sin(x))) \\ &= f(1 - 2\sin(x)) \\ &= (1 - 2\sin(x))^2 + 1 \\ &= 4\sin^2(x) - 4\sin(x) + 2.\end{aligned}$$

2 (9 points) Compute the derivatives of the following functions.

(a)

$$f(x) = 4x^3 + 7 + \frac{1}{x^3}$$

Solution.

$$f'(x) = 12x^2 - \frac{3}{x^4}.$$

(b)

$$f(x) = \sqrt{\tan^2(x) + 3}$$

Solution. We use the chain rule, twice.

$$\begin{aligned} f'(x) &= \frac{1}{2\sqrt{\tan^2(x)+3}} \left(\tan^2(x)+3\right)' \\ &= \frac{1}{2\sqrt{\tan^2(x)+3}} \, 2\tan(x)\big(\tan(x)\big)' \\ &= \frac{1}{2\sqrt{\tan^2(x)+3}} \, 2\tan(x)\sec^2(x) \\ &= \frac{1}{\sqrt{\tan^2(x)+3}} \, \tan(x)\sec^2(x). \end{aligned}$$

(c)

$$f(x) = e^x \ln(x) + \frac{e^x}{x^2+1}$$

Solution. We use the product rule and the quotient rule.

$$f'(x) = e^x \frac{1}{x} + e^x \ln(x) + \frac{e^x(x^2+1) - e^x(2x)}{(x^2+1)^2}.$$

3 (12 points)

(a) Compute the limit or explain why it does not exist.

$$\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3}$$

Solution.

$$\begin{aligned}\lim_{x\to -3} \frac{x^2-9}{x^2+2x-3} &= \lim_{x\to -3} \frac{(x+3)(x-3)}{(x+3)(x-1)} \\ &= \lim_{x\to -3} \frac{(x-3)}{(x-1)} \\ &= \frac{-6}{-4} \\ &= \frac{3}{2}.\end{aligned}$$

(b) Compute the limit or explain why it does not exist.

$$\lim_{x\to \infty} \frac{2+x+10x^3-3x^4}{x^2+2x^4}$$

Solution.

$$\begin{aligned}\lim_{x\to \infty} \frac{2+x+10x^3-3x^4}{x^2+2x^4} &= \lim_{x\to \infty} \frac{2/x^4+1/x^3+10/x-3}{1/x^2+2} \\ &= \frac{-3}{2}.\end{aligned}$$

(c) Compute the limit or explain why it does not exist.

$$\lim_{x \to 0} \cos(1/x^2)$$

Solution. This limit does not exist. As x approaches 0 from either side, the function $\cos(1/x^2)$ oscillates faster and faster between -1 and 1. A similar function is analyzed and graphed on page 98 of Stewart's book.

(d) Compute the limit or explain why it does not exist.

$$\lim_{x \to 0} x^2 \cos(1/x^2)$$

Solution. Note that

$$-x^2 \leq x^2 \cos(1/x^2) \leq x^2.$$

Since

$$\lim_{x \to 0} -x^2 = 0 = \lim_{x \to 0} x^2,$$

by the Squeeze Theorem on page 110 of Stewart we have

$$\lim_{x \to 0} x^2 \cos(1/x^2) = 0.$$

4 (12 points)

(a) If 1200 cm^2 of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

Solution. Let b be the side length of the square base, and let h be the height of the box. Hence the surface area of the box is $b^2 + 4bh$. Since the surface area of the box is also 1200 cm^2, we have

$$\begin{aligned} 1200 = b^2 + 4bh &\implies 1200 - b^2 = 4bh \\ &\implies \frac{1200 - b^2}{4b} = h. \end{aligned}$$

Hence the volume of the box is

$$V = b^2 h = b^2 \, \frac{1200 - b^2}{4b} = 300b - \frac{b^3}{4}.$$

To maximize the volume, we take the derivative and set it equal to zero. We get

$$0 = V' = 300 - \frac{3b^2}{4}$$

which implies

$$\frac{3b^2}{4} = 300 \implies b^2 = 400 \implies b = 20.$$

One can check that this is indeed a maximum by using the second derivative test, for instance. We plub $b = 20$ into the formula $V = 300b - \frac{b^3}{4}$ to see that the largest possible volume of the box is

$$300(20) - \frac{(20)^3}{4} = 6000 - \frac{8000}{4} = 4000 \text{ cm}^3.$$

(b) Suppose y is a function of x and that $\frac{dy}{dx} = x^2 y$. If $y(0) = -5$, find $y(2)$.

Solution. This is a separable differential equation. When $y \neq 0$ we can write

$$\frac{dy}{y} = x^2 \, dx$$

and integrate to get

$$\ln |y| = x^3/3 + C.$$

We take the exponential of each side to get

$$|y| = e^C e^{x^3/3} \Longrightarrow y = \pm e^C e^{x^3/3}.$$

We can write this more succinctly as

$$y = Ae^{x^3/3},$$

where A is an arbitrary constant (note when $A = 0$ we get the constant solution $y = 0$). We plug in the initial condition $y(0) = -5$ to see

$$-5 = Ae^{(0)^3/5} \Longrightarrow A = -5.$$

Hence

$$y = -5e^{x^3/3}$$

and

$$y(2) = -5e^{(2)^3/5} = -5e^{8/3}.$$

5 (10 points) Consider the function

$$f(x) = x^3 + 3x^2 + 1.$$

(a) (6 points) Sketch the graph of f using calculus techniques. Hint: Where are the local maxima and minima of f? Where is f concave up and concave down?

Solution. The first derivative of f is

$$f'(x) = 3x^2 + 6x = 3x(x+2).$$

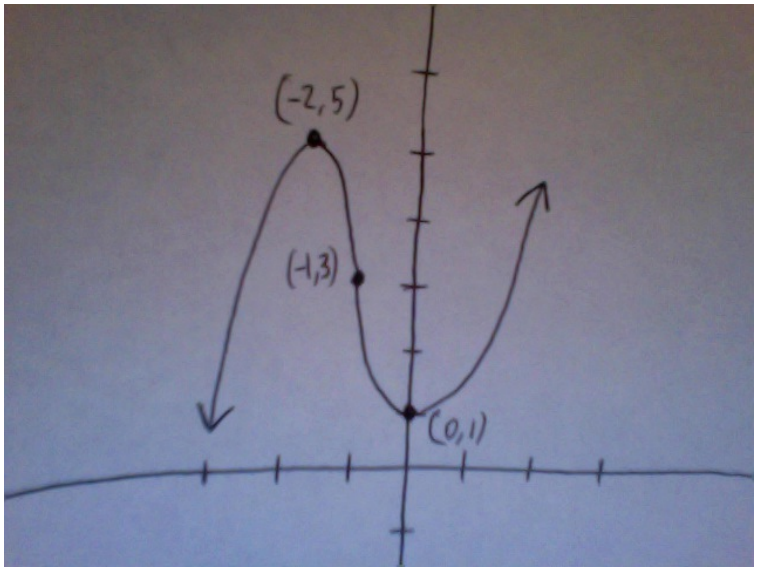
So the derivative is zero when $x = -2$ or 0, the derivative is positive when $x < -2$ or $x > 0$, and the derivative is negative when $-2 < x < 0$. So we have a local maximum at $x = -2$ and a local minimum at $x = 0$.

The second derivative of f is

$$f''(x) = 6x + 6 = 6(x+1).$$

Hence function f is concave down when $x < -1$, the function is concave up when $x > -1$, and we have an inflection point at $x = -1$.

We compute $f(-2) = 5$, $f(0) = 1$, and $f(-1) = 3$. We plot these three points and then sketch the graph of f using the derivative and concavity information that we have determined above.



(b) (4 points) Estimate $f(0.99)$ by using a linear approximation to $f(x)$ at $x = 1$. This is also called a tangent line approximation or a linearization.

Solution. The linear approximation formula is

$$f(x) \approx f(a) + f'(a)(x - a),$$

for x near a. See page 241 of Stewart. We apply this formula with $x = 0.99$ and $a = 1$. Note that

$$f(1) = (1)^3 + 3(1)^2 + 1 = 5$$

and

$$f'(1) = 3(1)^2 + 6(1) = 9.$$

We get

$$\begin{aligned} f(0.99) &\approx f(1) + f'(1)(0.99 - 1) \\ &= 5 + 9(-.01) \\ &= 4.91. \end{aligned}$$

6 (12 points)

(a) Evaluate the integral or explain why it diverges.

$$\int x^3 + 2 \, dx$$

Solution.

$$\int x^3 + 2 \, dx = \frac{1}{4}x^4 + 2x + C.$$

(b) Evaluate the integral or explain why it diverges.

$$\int x \sin(x) \, dx$$

Solution. We use the integration by parts formula

$$\int u \, dv = uv - \int v \, du.$$

with $u = x$ and $dv = \sin(x) \, dx$. Note $du = dx$ and $v = -\cos(x)$. We get

$$\begin{aligned}\int x \sin(x) \, dx &= x\big(-\cos(x)\big) - \int \big(-\cos(x)\big) \, dx \\ &= -x\cos(x) + \int \cos(x) \, dx \\ &= -x\cos(x) + \sin(x) + C.\end{aligned}$$

(c) Evaluate the integral or explain why it diverges.

$$\int_0^2 xe^{x^2+3}\,dx$$

Solution. We use the technique of substitution. Let $u = x^2 + 3$. Note that $du = 2x\,dx$. We must also be careful with the limits of integration. As x varies from 0 to 2, $u = x^2 + 3$ varies from 3 to 7. Hence

$$\begin{aligned}\int_0^2 xe^{x^2+3}\,dx &= \frac{1}{2}\int_0^2 2xe^{x^2+3}\,dx \\ &= \frac{1}{2}\int_3^7 e^u du \\ &= \frac{1}{2}e^u\Big|_3^7 \\ &= \frac{1}{2}(e^7 - e^3).\end{aligned}$$

(d) Evaluate the integral or explain why it diverges.

$$\int_0^3 \frac{1}{x-1}\,dx$$

Solution. Since the function $\frac{1}{x-1}$ is not continuous at $x = 1$, this is an improper integral. We need to split it into two parts.

$$\int_0^3 \frac{1}{x-1}\,dx = \int_0^1 \frac{1}{x-1}\,dx + \int_1^3 \frac{1}{x-1}\,dx.$$

Let's analyze the two parts separately. For the first part, we have

$$\begin{aligned}\int_0^1 \frac{1}{x-1}\,dx &= \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx \\ &= \lim_{t\to 1^-}\Big(\ln|x-1|\Big|_0^t\Big) \\ &= \lim_{t\to 1^-}\big(\ln|t-1| - \ln|-1|\big) \\ &= \lim_{t\to 1^-}\big(\ln|t-1| - e\big) \\ &= -\infty.\end{aligned}$$

Since this first part diverges, we know already that the entire integral

$$\int_0^3 \frac{1}{x-1}\,dx$$

diverges, and we do not need to analyze the second part at all.

7 (6 points)

(a) Determine whether the series is convergent or divergent. If it is convergent, find the sum.

$$4 - 3 + \frac{9}{4} - \frac{27}{16} + \frac{81}{64} - \dots$$

Solution. This is a geometric series with ratio $-3/4$ and first term 4. Since $|-3/4| < 1$, the series is convergent and its sum is

$$\frac{4}{1-(-3/4)} = \frac{4}{7/4} = \frac{16}{7}.$$

See page 567 of Stewart.

(b) Determine whether the series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{n^2-1}{n^2}$$

Solution. Note that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2} = \lim_{n\to\infty} = \lim_{n\to\infty} 1 - 1/n^2 = 1 \neq 0.$$

Since the terms of this series do not tend to zero, this series is divergent. See page 570 of Stewart.