Selected solutions for SSEA 51, Homework 1

LA 3.8. Let $\{u, v, w\}$ be a linearly independent set. Is

$$\{u - v, v - w, u - w\}$$

a linearly independent set? Show that it is or show why it is not.

Solution. Since $\{u, v, w\}$ is linearly independent, we know that if we have an equation of the form

$$c_1 u + c_2 v + c_3 w = \vec{0},$$

then necessarily $0 = c_1 = c_2 = c_3$. This is our given information.

Let's find out whether $\{u - v, v - w, u - w\}$ is linearly independent or not. To do so, we consider the equation

$$d_1(u - v) + d_2(v - w) + d_3(u - w) = \vec{0}.$$

Must all the coefficients be zero? Let's regroup the terms according to the vectors u, v, and w. We get

$$(d_1 + d_3)u + (-d_1 + d_2)v + (-d_2 - d_3)w = \vec{0}.$$

We did this regrouping because now we can now apply the given information. Since $\{u, v, w\}$ is linearly independent, the coefficients above must all be zero. That is, we have

$$\begin{aligned} d_1 + d_3 &= 0 \\ -d_1 + d_2 &= 0 \\ -d_2 - d_3 &= 0. \end{aligned}$$

The first equation gives $d_1 = -d_3$. We plug into the second equation to get $d_3 + d_2 = 0$, or $d_2 = -d_3$. This is also the same as the third equation. Note we can pick d_3 to be any real number and still solve this system for d_1 and d_2. Let's pick d_3 to be nonzero, say $d_3 = 1$. Then we get $d_1 = -1$ and $d_2 = -1$.

We plug this back into the equation

$$d_1(u - v) + d_2(v - w) + d_3(u - w) = \vec{0}$$

to see that

$$-1(u - v) - 1(v - w) + 1(u - w) = \vec{0}.$$

Hence the set $\{u - v, v - w, u - w\}$ is linearly dependent.

Remark: if you were able to see the linear dependency

$$-1(u - v) - 1(v - w) + 1(u - w) = \vec{0}$$

just by staring at the vectors, then that's fine too. But recognize that this approach won't work when the vectors happen to be linearly independent.

LA 3.10. If $S = \{v_1, \ldots, v_k\}$ is a set of linearly independent vectors in $\mathbb{R}^n$, then any subset of S must be linearly independent.

Solution. This is true. Let's prove it.

Suppose $S = \{v_1, \ldots, v_k\}$ is linearly independent. This means that if we have an equation of the form

$$c_1 v_1 + \ldots + c_k v_k = \vec{0},$$

then necessarily $0 = c_1 = \ldots = c_k$. This is our given information.

Now, suppose we have a subset of S of size $m < k$. Without loss of generality, let's relabel the vectors in this subset to be $\{v_1, \ldots, v_m\}$. We need to show that $\{v_1, \ldots, v_m\}$ is linearly independent, and so we consider the equation

$$d_1 v_1 + \ldots + d_m v_m = \vec{0}.$$

Note that we can add $0v_{m+1} + \ldots + 0v_k = \vec{0}$ without changing anything. So we also have

$$d_1 v_1 + \ldots + d_m v_m + 0v_{m+1} + \ldots + 0v_k = \vec{0}.$$

But now this is in the same form as our given information. Since $\{v_1, \ldots, v_k\}$ is linearly independent, all the coefficients above must be zero (including the last few coefficients that we already knew were zero). That is, necessarily $0 = d_1 = \ldots d_m = 0$. So we have shown that $\{v_1, \ldots, v_m\}$ is linearly independent. Hence any subset of S is linearly independent.

LA 3.12. If $\text{span}(v_1, v_2, v_3) = \mathbb{R}^3$, then $\{v_1, v_2, v_3\}$ must be a linearly independent set.

Solution. Let's prove the contrapositive, which is the same as proving this statement. That is, we'll prove that if $\{v_1, v_2, v_3\}$ is linearly dependent then $\text{span}(v_1, v_2, v_3)$ is not all of $\mathbb{R}^3$.

Suppose $\{v_1, v_2, v_3\}$ is linearly dependent. Then by definition, at least one of the vectors v_1, v_2, or v_3 is a linear combination of the other two. Without loss of generality, let's relabel the vectors so that v_3 is a linear combination of v_1 and v_2. That is, $v_3 = c_1 v_1 + c_2 v_2$ for some $c_1, c_2 \in \mathbb{R}$.

You can use the equation $v_3 = c_1 v_1 + c_2 v_2$ to show that $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2)$. I haven't written out all the details of why this is true, but I will if you ask me. The span of two vectors is either a point, a line, or a plane, and hence not all of $\mathbb{R}^3$. Hence $\text{span}(v_1, v_2, v_3) = \text{span}(v_1, v_2)$ is not all of $\mathbb{R}^3$. This proves the contrapositive, which is the same as proving the original statement.