Challenge Problems for SSEA 41, Homework 3

The first four problems concern polynomial approximations to functions.

Let $f(x)$ be a function. If we want to approximate f as closely as possible near $x = a$ with a line, then we should choose then tangent line. Let $g(x)$ be this tangent line, or linear approximation, to $f(x)$ at $x = a$. Note that we have $g(a) = f(a)$ and $g'(a) = f'(a)$. That is, the functions and their first derivatives agree at $x = a$.

Suppose now that we want to approximate f as closely as possible near $x = a$ with a quadratic polynomial. Let $g(x)$ be this quadratic polynomial. It turns out that if $g(a) = f(a)$, if $g'(a) = f'(a)$, and if $g''(a) = f''(a)$, then $g(x)$ is the best quadratic approximation to $f(x)$ at $x = a$. That is, the functions and their first two derivatives agree at $x = a$.

(1) What quadratic polynomial most closely approximates the function $f(x) = e^x$ near $x = 0$?

(2) What quadratic polynomial most closely approximates $f(x) = e^{-x}$ near $x = 2$? It may help to write the polynomial as powers of $(x - 2)$ rather than as powers of x.

(3) What cubic polynomial most closely approximates $f(x) = e^x$ near $x = 0$? Guess a general formula for the best nth degree polynomial approximation for $f(x) = e^x$.

(4) Given a function $f(x)$ that is n times differentiable at $x = a$, find a general formula for the nth degree polynomial that most closely approximates $f(x)$ near $x = a$. Your answer should depend on various derivatives of $f(x)$ at $x = a$.

(5) Tie three pieces of string into three loops (or draw a picture of the loops) so that:
   (a) The three loops are linked. That is, you can't pull one loop arbitrarily far away from the others.
   (b) After cutting *any* one of the three loops, the remaining two loops are not linked. That is, you can pull the two remaining loops arbitrarily far apart.