

Challenge Problems for SSEA 41, Homework 2

The first three problems concern *complex numbers*. Recall that i is a complex number that represents a square root of -1 , so that $i^2 = -1$. When taking derivatives and performing other algebraic manipulations, i is treated like any other constant.

- (1) Consider the complex exponential function $f(\theta) = e^{i\theta}$. Take its first and second derivatives. What is the relationship between $f''(\theta)$ and $f(\theta)$?
- (2) Let $g(\theta) = \cos \theta + i \sin \theta$. What is the relationship between $g''(\theta)$ and $g(\theta)$?
- (3) An identity that is important in applications to physics is

$$e^{i\theta} = \cos \theta + i \sin \theta.$$

Note that because of this identity, (1) implies (2) and vice versa. This identity expresses a complex exponential function in terms of trigonometric functions. Can you find a way of representing $\sin \theta$ and $\cos \theta$ in terms of complex exponential functions?

(Hint: it may help to think about what happens when you replace θ with $-\theta$ in the identity.)

- (4) Consider the differential equation

$$f'(x) = f(x).$$

A function f satisfies this equation when it is equal to its own derivative. As you know, $f(x) = e^x$ is such a function.

It is a fact that if f satisfies $f'(x) = f(x)$ and has $f(0) = 0$, then $f(x) = 0$ for all x . Using this fact, prove that e^x is the ONLY function with $f(0) = 1$ that satisfies $f'(x) = f(x)$.

(Hint: if $g(x)$ is another such function, consider $f(x) - g(x)$.)

- (5) If you have an 8×8 square board containing 64 total squares, then you can cover it with 32 tiles of size 2×1 . If you remove the top right and bottom left squares from the board, then you have 62 squares remaining. Can you cover the remaining 62 squares with 31 tiles of size 2×1 ? It turns out that you can't do it (but you should try)! Give a short and simple yet clever explanation for why it is impossible to cover the remaining squares. Your explanation should be simple enough to convince a middle schooler in two minutes. If you are stuck, ask Henry for a hint.