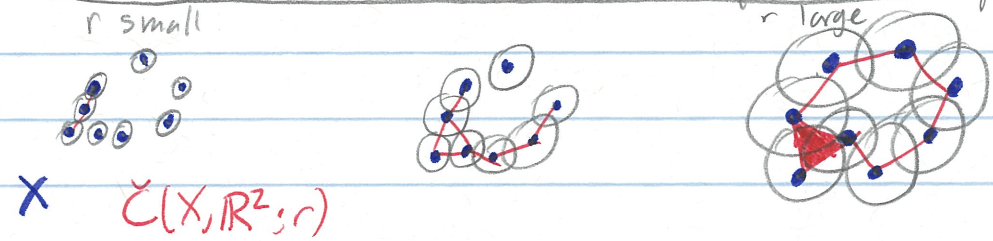
Čech and Vietoris-Rips simplicial complexes



Let Y be a metric space (often $Y = \mathbb{R}^n$).
For $x \in Y$, let $B(x,r) = \{ y \in Y \mid d(y,x) \leq r \}$.

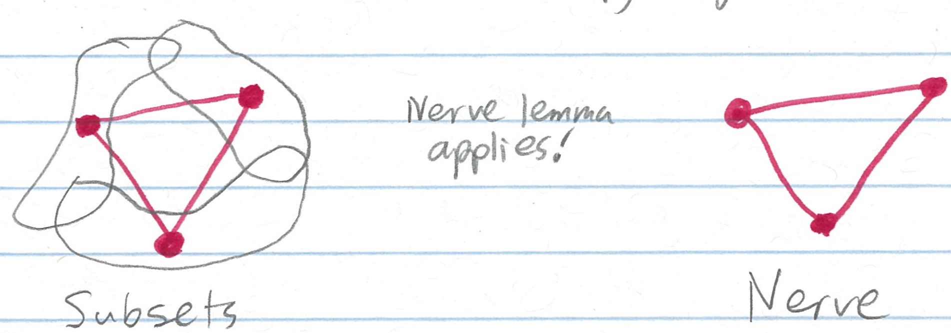
Def For $X \subseteq Y$ and $r \geq 0$, the <u>Čech simplicial complex</u> $\check{C}(X, Y; r)$ (or $\check{C}(X; r)$) has

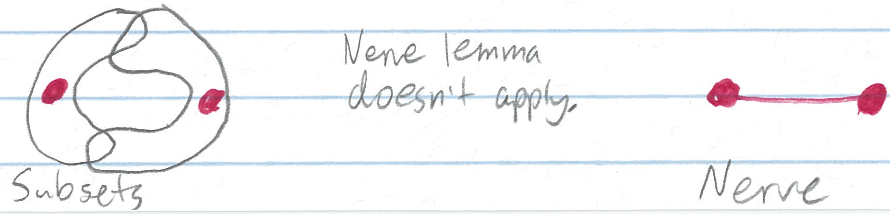
- vertex set X
- a simplex $[x_0, x_1, \ldots, x_k]$ if $\bigcap_{i=0}^{k} B(x_i, r) \neq \emptyset$.

Thm $\check{C}(X, \mathbb{R}^n; r) \simeq \bigcup_{x \in X} B(x, r)$

This follows from the Nerve Lemma, which says that if you have a collection of contractible subsets s.t. each intersection of subsets is empty or contractible, then the <u>nerve</u> of the subsets is homotopy equivalent to the union.

Pic

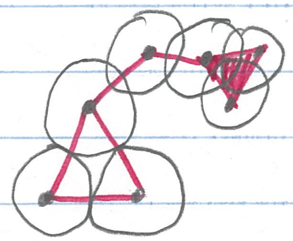




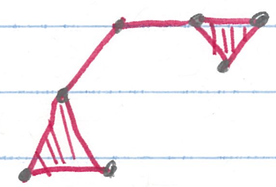
Čech complexes are theoretically very nice, but hard to compute in high-dimensional space.

<u>Def</u> For X a metric space and $r \geq 0$, the <u>Vietoris-Rips simplicial complex</u> $VR(X;r)$ has

- vertex set X
- a simplex $[x_0, \ldots, x_k]$ if $d(x_i, x_j) \leq r$ $\forall i,j$.



$\check{C}(X;r)$



$VR(X;2r)$

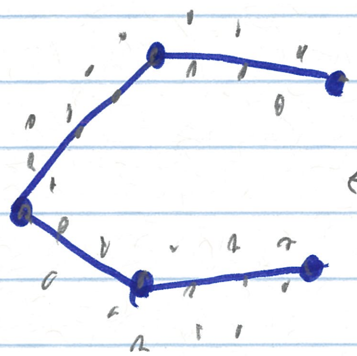
<u>Prop</u> $\check{C}(X, \mathbb{R}^n; r) \subseteq VR(X;2r) \subseteq \check{C}(X, \mathbb{R}^n; 2r)$

<u>Thm (Hausmann '95)</u> For M a Riemannian manifold and r sufficiently small, $VR(M;r) \simeq M$

<u>Thm (Latscher '01)</u> For M a Riemannian manifold, r sufficiently small, and $X \subseteq M$ sufficiently dense, $VR(X;r) \simeq M$

Witness complexes

The main idea is to only select a subset of data points as the vertex set, and to use the presence of other data points to "witness" the appearance of higher-dimensional simplices.

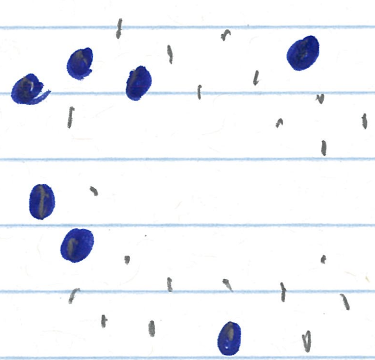


← no edge!

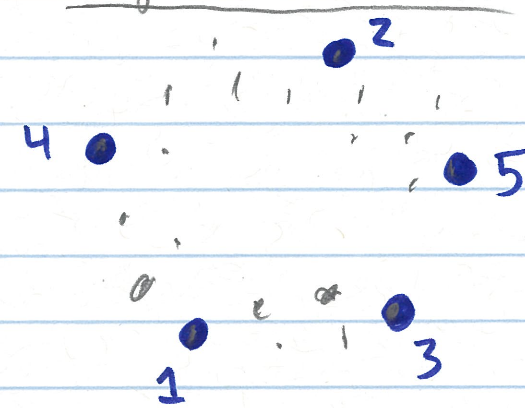
Witness complex on 5 vertices

How to choose the vertices (or "landmarks")?

Random vertices



Sequential maxmin



"Covers" the dataset <u>but</u> selects outliers.