CSU Math 571
Spring 2018

Homework 4

Due Friday, February 9 at the beginning of class

Reading. Section 1.3 of Hatcher

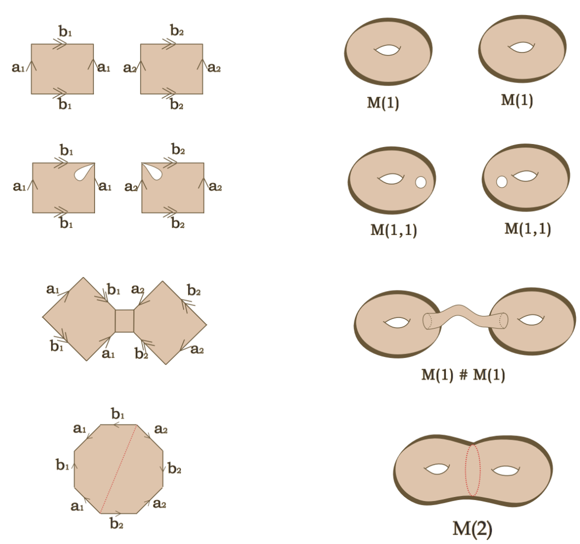
Problems.

1. (8 points) The classification of closed surfaces ([1]) says that a connected closed surface (where "closed" here means compact with no boundary) is homeomorphic to exactly one of the following:

   - the sphere $M_0 := S^2$,
   - the connected sum of g tori for $g \geq 1$, denoted M_g, and also called the torus of genus g, or
   - the connected sum of $g \geq 1$ projective planes for $g \geq 1$, denoted N_g.

   The connected sum of two surfaces is obtained by deleting a disk from each, and then gluing the two surfaces together along their two boundary circles. It turns out that the M_g surfaces are orientable, whereas the N_g surfaces are not.

   (a) Show that M_g has a CW complex structure with one 0-cell, $2g$ 1-cells, and one 2-cell, and deduce from this CW structure that the fundamental group of M_g is $\pi_1(M_g) \cong \langle a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle$. Hatcher has pictures for $g = 1, 2, 3$ on page 5, but the picture below may be more helpful.



[1]https://en.wikipedia.org/wiki/Surface_(topology)#Classification_of_closed_surfaces

(b) Show that N_g has a CW complex structure with one 0-cell, g 1-cells, and one 2-cell, and deduce from this CW structure that the fundamental group of N_g is $\pi_1(N_g) \cong \langle a_1, \ldots, a_g \mid a_1^2 \cdots a_g^2 \rangle$.

(c) "Abelianization" is a way to turn any group into an ableian group. See ([2]) or ([3]). Indeed, there is a functor Ab: Grp $\to$ AbGrp (called abelianization) from the category of groups to the category of abelian groups. Compute the abelianizations of $\pi_1(M_g)$ and $\pi_1(N_g)$.

(d) Conclude that none of the connected closed surfaces M_g for $g \geq 0$ or N_g for $g \geq 1$ are homeomorphic (or even homotopy equivalent) to each other.

Remark: Hatcher talks about this on pages 51-52; the point here is to learn all of the details.

2. (6 points) Hatcher Exercise 9 on page 79: Show that if a path-connected, locally path-connected space X has $\pi_1(X)$ finite, then every map $X \to S^1$ is nullhomotopic.

3. (6 points) Hatcher Exercise 10 on page 79: Find all the connected 2-sheeted and 3-sheeted covering spaces of $S^1 \vee S^1$ up to isomorphism of covering spaces without basepoints (defined on page 67). You do not need to prove your answer is correct.

Challenge Problem (No credit or extra credit available). Do Exercise 12 on page 80 of Hatcher: Let a and b be the generators of $\pi_1(S^1 \vee S^1)$ corresponding to the two S^1 summands. Draw a picture of the covering space of $S^1 \vee S^1$ corresponding to the normal subgroup generated by a^2, b^2, and $(ab)^4$, and prove that this covering space is indeed the correct one.

Remark: The normal subgroup generated by a^2, b^2, and $(ab)^4$ is larger than the group $\langle a^2, b^2, (ab)^4 \rangle$. It's helpful to read what a normal covering space is before attempting this problem.

[2] `http://mathworld.wolfram.com/Abelianization.html`

[3] `https://groupprops.subwiki.org/wiki/Abelianization#Abelianization_as_a_group`