CSU Math 570
Fall 2017

Homework 7

Due Friday, October 13 at the beginning of class

Reading.
Finish reading Chapter 7.

Problems.

1. Suppose $f, g\colon S^n \to S^n$ are continuous maps such that $f(x) \neq -g(x)$ for any $x \in S^n$. Prove that f and g are homotopic.

   Hint: As an easier case, what if we instead had $f, g\colon S^n \to \mathbb{R}^{n+1} \setminus \{\vec{0}\}$ with the the line segment from $f(x)$ to $g(x)$ not passing through the origin for all $x \in S^n$? Can you modify a proof of this easier case to handle the problem above where $f, g\colon S^n \to S^n$ with $f(x) \neq -g(x)$?

2. Let X be a topological space and let g be a path in X from p to q. Let $\Phi_g\colon \pi_1(X,p) \to \pi_1(X,q)$ denote the group isomorphism defined in Theorem 7.13.

   If $h\colon X \to Y$ is continuous, then show that the following diagram commutes:

$$\begin{tikzcd} \pi_1(X,p) \arrow[r, "h_*"] \arrow[d, "\Phi_g"] & \pi_1(Y,h(p)) \arrow[d, "\Phi_{h\circ g}"] \\ \pi_1(X,q) \arrow[r, "h_*"] & \pi_1(Y,h(q)) \end{tikzcd}$$

3. Let X be a path-connected topological space, and let $p, q \in X$. Show that $\pi_1(X,p)$ is abelian if and only if for any two paths g, g' from p to q in X, we have $\Phi_g = \Phi_{g'}$ (as isomorphisms from $\pi_1(X,p)$ to $\pi_1(X,q)$).

4. Let $F\colon C \to D$ be a (covariant) functor from category C to category D. Prove that if $X, Y \in \mathrm{Obj}(C)$ are isomorphic objects in C, then $F(X), F(Y)$ are isomorphic objects in D.

   Remark: This is essentially Theorem 7.51 in our book; I'm not asking you to quote this theorem.

5. Let X be a toplogical space. Prove that the following statements are equivalent:

   (i) X is compact.

   (ii) For every collection $\{C_\alpha\}_{\alpha\in A}$ of closed subsets of X with $\cap_{\alpha\in A} C_\alpha = \emptyset$, there is a finite subcollection $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ with $C_{\alpha_1} \cap \ldots \cap C_{\alpha_n} = \emptyset$.