CSU Math 570
Fall 2017

Homework 5

Due Friday, September 22 at the beginning of class

Reading.
Read Chapter 4 through page 98 and Chapter 7 through page 200.

Problems.

1. (a) Prove that a topological space X is disconnected if and only if there exists a surjective continuous function from X to the discrete space $\{0, 1\}$.
   (b) Prove that if X is path-connected and $f: X \to Y$ is continuous, then $f(X)$ is path-connected.

2. Prove Lemma 4.27 in our book, which says that if X is a topological space, then $A \subseteq X$ (with the subspace topology) is compact if and only if every cover of A by open subsets of X has a finite subcover.

3. Solutions to this problem are in our book — feel free to learn and use those solutions!
   (a) Let X be a Hausdorff space and let $A, B \subseteq X$ be disjoint compact subsets. Prove that there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$.
   (b) Prove that every compact subset A of a Hausdorff space X is closed.

4. Define $id: S^1 \to S^1$ by $id(p) = p$, and define $g: S^1 \to S^1$ by $g(p) = -p$. Find a homotopy $F: S^1 \times I \to S^1$ from id to g.

   Remark: It's perhaps easiest to represent S^1 using complex coordinates, $S^1 = \{e^{i\theta}\} \subseteq \mathbb{C}$, where θ varies between 0 and 2π. Then $id(e^{i\theta}) = e^{i\theta}$ and $g(e^{i\theta}) = -e^{i\theta} = e^{i(\theta+\pi)}$. For this problem, you can write down a map and say "this map is clearly continuous" without verifying that fact (so long as it's true).

5. Let $n > 1$. Prove that $\mathbb{R}^n$ is not homeomorphic to any open subset of $\mathbb{R}$.

6. Let A be an infinite set, and let $\mathbb{R}^A$ denote the Cartesian product of A copies of $\mathbb{R}$ (namely $\mathbb{R}^A = \prod_{a \in A} X_\alpha$ where $X_\alpha = \mathbb{R}$ for all $\alpha \in A$). Consider $\mathbb{R}^A$ equipped with two different toplogies: $(\mathbb{R}^A, \text{product})$ with the product topology, and $(\mathbb{R}^A, \text{box})$ with the box topology, as defined on page 63 of our book.

   Show that $(\mathbb{R}^A, \text{box})$ equipped with the maps $\pi_\alpha^{box}: (\mathbb{R}^A, \text{box}) \to X_\alpha = \mathbb{R}$ defined via $\pi_\alpha^{box}((x_\alpha)_{\alpha \in A}) = x_\alpha$ is not the categorical product of A copies of $\mathbb{R}$ in the category of topological spaces, as follows (and *not* by using Corollary 3.39). Suppose for a contradiction $(\mathbb{R}^A, \text{box})$ satisfied the universal property on page 213. Choose W to be the actual categorical product $(\mathbb{R}^A, \text{product})$ equipped with the maps $\pi_\alpha^{prod}\colon (\mathbb{R}^A, \text{box}) \to X_\alpha = \mathbb{R}$ similarly defined via $\pi_\alpha^{prod}((x_\alpha)_{\alpha\in A}) = x_\alpha$. Show that there is no continuous f making the necessary diagrams commute.

Hint: Note that $(0,1)^A$ *is open in* $(\mathbb{R}^A, \mathit{box})$. *Can you explain why* $(0,1)^A$ *is not open in* $(\mathbb{R}^A, \mathit{product})$*?*

Remark: When showing that an object is not a categorical product, it is often a good idea to choose "test object" W *to be the actual categorical product.*